A Piece-wise Linear Model of Credit Traps and Credit Cycles: A Complete Characterization

By

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1. Introduction

- Matsuyama's (AER 2007) regime-switching model of credit frictions, where
    - Entrepreneurs have access to heterogeneous investment projects
    - Due to credit frictions, entrepreneurs' net worth affects their ability to finance different projects
    - A change in the current level of net worth causes credit flows to switch across projects with different productivity
    - This in turn affects the future level of net worth.

- It was shown that this model generates a rich array of dynamic behavior.
    - Credit Traps
    - Credit Cycles
    - Leapfrogging
    - Reversal of Fortune
    - Growth Miracle

- But, a complete characterization of the dynamic behavior was lacking.

- Here, we offer a complete characterization for Cobb-Douglas production function, which makes the dynamical system piece-wise linear.

2.A regime switching model of credit frictions: A quick review of Matsuyama (2007)

A Variation of the Diamond OG model

Final Good: $Y_t = F(K_t,L_t)$, with physical capital, K_t and labor, L_t.

$$y_t \equiv Y_t/L_t = F(K_t/L_t,1) \equiv f(k_t), \text{ where } k_t \equiv K_t/L_t;\ f'(k) > 0 > f''(k).$$

Competitive Factor Markets: $\rho_t = f'(k_t)$; $w_t = f(k_t) - k_t f'(k_t) \equiv W(k_t) > 0$.

Agents: A unit measure of homogeneous agents.

In the 1st period, they supply one unit of labor, earn and save $W(k_t)$.

In the 2nd period, they consume.

Their objective is to maximize the 2nd period consumption.

Investment Technologies: Agents can choose one (and only one) from J indivisible projects ($j = 1,2, \ldots J$).

	Period t		*Period t+1*
Project-j:	m_j units of final good	→	m_jR_j units of capital

m_j: the (fixed) set-up cost, R_j: the project productivity

To Invest or Not to Invest?

By starting a project-j, $C_t = m_jR_j\rho_{t+1} - r_{t+1}(m_j - w_t)$, $(j = 1,2,\ldots,J)$

By lending, $C_t = r_{t+1}w_t$

Profitability Constraint: $R_jf'(k_{t+1}) \geq r_{t+1}$, (*PC-j*)

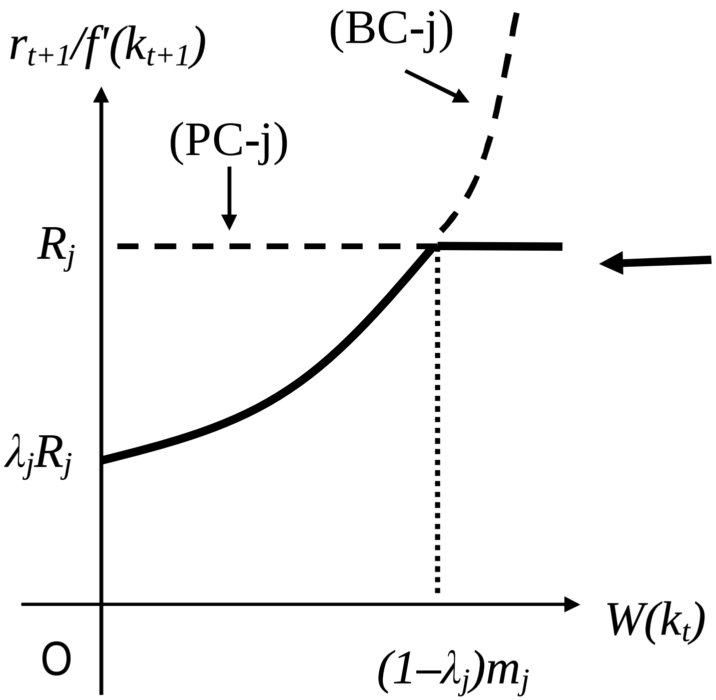
Credit Frictions (introduced by the pledgeability constraint a la Holmstrom-Tirole):

Borrowing Constraint: $\lambda_jm_jR_jf'(k_{t+1}) \geq r_{t+1}(m_j - W(k_t))$, (*BC-j*)

λ_j: the pledgeable fraction of the project revenue

Both (*PC-j*) and (*BC-j*) must be satisfied for the credit to flow into type-j projects.

What is the maximal rate of return the borrowers can pledge to the lenders from running a type–j project? From (PC-j) and (BC-j),



$$\frac{r_{t+1}}{f'(k_{t+1})} = \frac{R_j}{\max\{1, [1 - W(k_t)/m_j]/\lambda_j\}}$$

Equilibrium Conditions

(1) $W(k_t) = \sum_j (m_j X_{jt})$.

(2) $k_{t+1} = \sum_j (m_j R_j X_{jt})$.

(3) $$\frac{r_{t+1}}{f'(k_{t+1})} \geq \frac{R_j}{\max\{1, [1 - W(k_t)/m_j]/\lambda_j\}} \qquad (j = 1, 2, \ldots J)$$

where X_{jt} is the measure of type-j projects initiated in period t, and $X_{jt} > 0$ (j = 1, 2,…J) implies the equality in (3).

For each k_t, we can rank the projects by the RHS of (3). Thus, generically, only one type of project, $J(k_t)$, gets all the credit; $X_{jt} = 1$ if $j = J(k_t)$, and $X_{jt} = 0$, otherwise.

Hence, we call this "a regime-switching" model.

This means that eqs. (1)-(3) are simplified to:

(4) $$k_{t+1} = R_{J(k_t)} W(k_t).$$

For $k_0 > 0$, (4) determines the equilibrium trajectory. ($W(k_t)$ is assumed to be concave, so that the dynamics would be simple without regime-switching.)

With the Cobb-Douglas production, $y_t = A(k_t)^\alpha$ with $0 < \alpha < 1$, eq. (4) can be rewritten as a PWL system with a regime-dependent constant term:

(5) $$x_{t+1} = \theta_{\hat{J}(x_t)} + \alpha x_t$$

by defining,

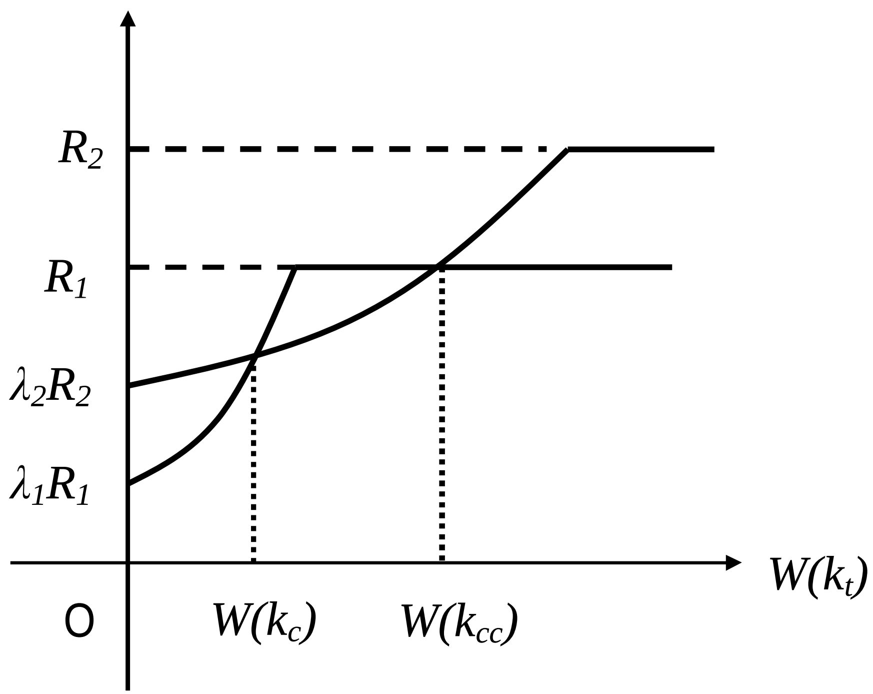
$$x_t \equiv \log_b k_t; \qquad \theta_{\hat{J}(x_t)} \equiv \log_b (1-\alpha) A R_{J(k_t)}$$

where $\hat{J}(x_t) \equiv J(k_t)$ and b is the base of the logarithm.

Below, let us focus on the following case considered in Matsuyama (2007, Sec.4).

$$R_2 > R_1 > \lambda_2 R_2 > \lambda_1 R_1, \text{ and } m_2/m_1 > (1-\lambda_1)/(1-\lambda_2 R_2/R_1).$$

- Project 1 is less productive and less pledgeable than Project 2.
- Project 1 requires the smaller set-up cost than Project 2.



$$(8) \qquad k_{t+1} = \begin{cases} R_2 W(k_t) & \text{if} \quad k_t < k_c \text{ or } k_t > k_{cc} \\ R_1 W(k_t) & \text{if} \quad k_c < k_t < k_{cc}. \end{cases}$$

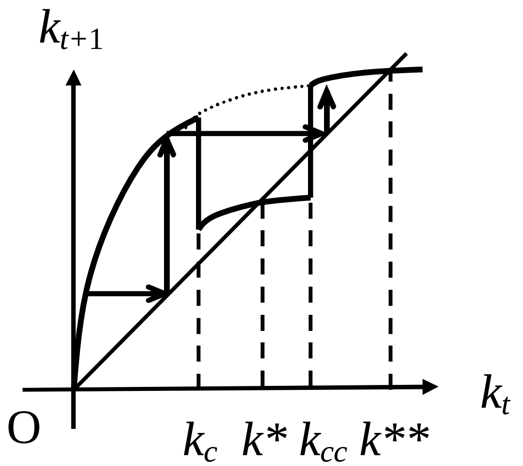


Figure 5a

Credit Trap
or Leapfrogging
or Reversal of Fortune

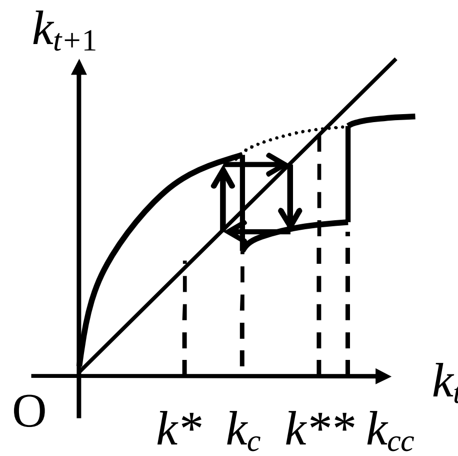


Figure 5b

Credit Cycles

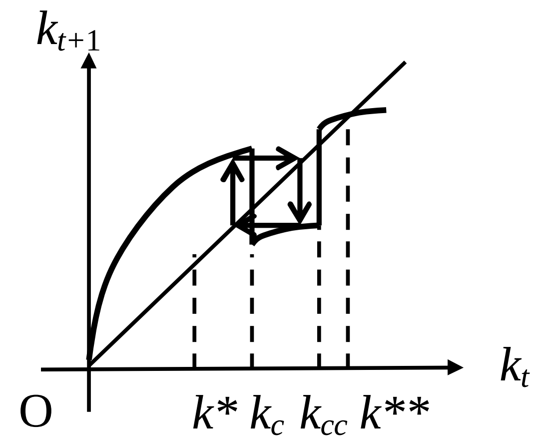


Figure 5c

Cycles as a Trap
or Growth Miracle

For the Cobb-Douglas Production; $y_t = A(k_t)^{\alpha}$ with $0 < \alpha < 1$:

$$x_{t+1} = \begin{cases} \theta_2 + \alpha x_t & \text{if} \quad x_t \le d_1 \text{ and } x_t \ge d_2 \\ \theta_1 + \alpha x_t & \text{if} \quad d_1 < x_t < d_2 \end{cases}$$

by defining,

$$x_t \equiv \log_b k_t;$$
$$\theta_1 \equiv \log_b R_1(1-\alpha)A < \theta_2 \equiv \log_b R_2(1-\alpha)A;$$
$$d_1 \equiv \log_b(k_c) < d_2 \equiv \log_b(k_{cc}),$$

Three goods (final, capital, labor) → Two degrees of freedom in choosing units of measurement. We set

$$A = 1/R_1(1-\alpha) \qquad \text{and} \qquad b = \left(\frac{R_2}{R_1}\right)^{1/(1-\alpha)},$$

so that $\theta_1 = 0$ and $\theta_2 = 1-\alpha$. Then,

$$x_{t+1} = f(x_t) = \begin{cases} f_L(x_t) \equiv (1-\alpha) + \alpha x_t & \text{if} \quad x_t \le d_1 \text{and } x_t \ge d_2 \\ f_R(x_t) \equiv \alpha x_t & \text{if} \quad d_1 < x_t < d_2 \end{cases}$$

with the two discontinuity (or switching) points, $d_1 < d_2$, given by:

$$\alpha d_1 \equiv \log_b \left(\frac{(\lambda_2 / \lambda_1)(R_2 / R_1) - 1}{(\lambda_2 / \lambda_1)(R_2 / R_1)(m_2 / m_1) - 1} \right) (R_1 m_2);$$

$$\alpha d_2 \equiv \log_b \left(1 - \lambda_2 (R_2 / R_1)\right)(R_1 m_2)$$

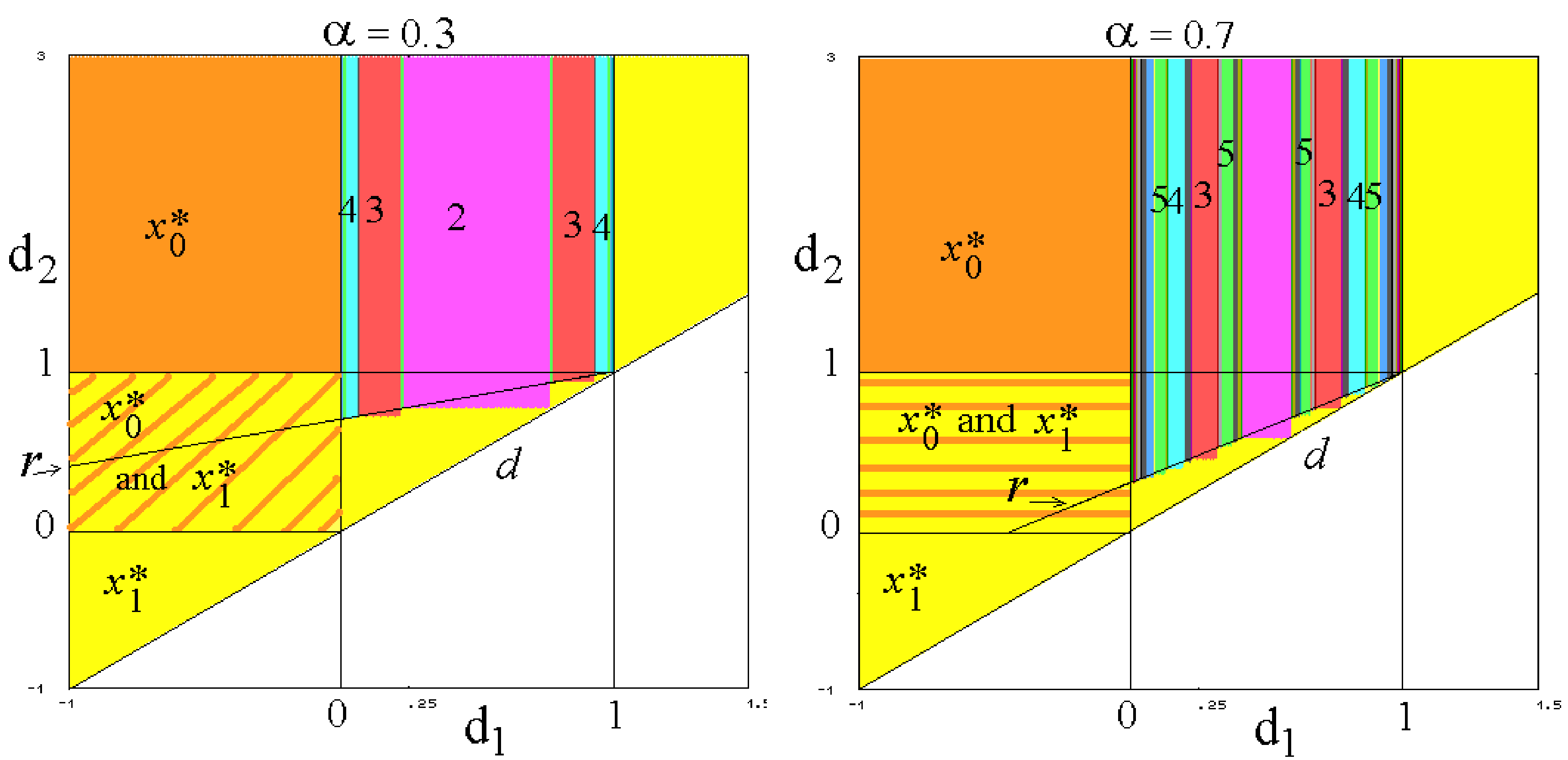
Note:

- If the credit always flowed to the less productive type-1 projects,
  $x_{t+1} = f_R(x_t) \equiv \alpha x_t$, converging monotonically to $x_R^* = 0$.
- If the credit always flowed to the more productive type-2 projects,
  $x_{t+1} = f_L(x_t) \equiv (1-\alpha) + \alpha x_t$, converging monotonically to $x_L^* = 1$.
- Credit friction parameters, $(\lambda_1, \lambda_2, m_1, m_2)$, affect dynamics through (d_1, d_2).

Let us see how this PWL system changes with (d_1, d_2).

4. Analysis

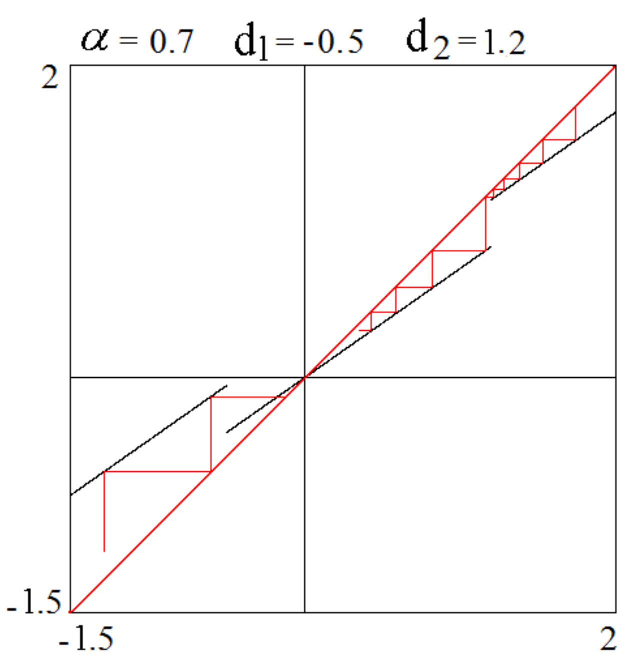
A Preview of the Results in the parameter space, $(d_1 < d_2)$, for $\alpha = 0.3$ and $\alpha = 0.7$

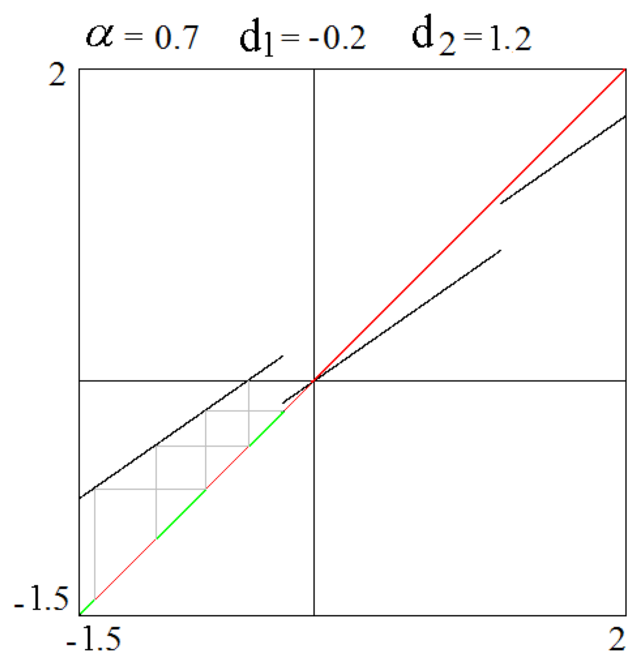


Three Relatively Simple Cases: In all these cases, the dynamics ***globally converges*** to its unique steady state and the equilibrium trajectory changes the direction ***at most once***.

Case S-I: ($d_1 < 0$ & $d_2 > 1$), *Orange*; convergent to its unique steady state, $x_R^* = 0$.

- If $\alpha d_1 + 1 - \alpha < 0$, monotone increasing for $x_0 \in (-\infty, 0)$ and monotone decreasing for $x_0 \in (0, \infty)$, as shown in the left Figure.
- If $\alpha d_1 + 1 - \alpha > 0$, monotone increasing for $x_0 \in \cup_{n \geq 0} f_L^{-n}(d_1, 0)$ & monotone decreasing for $x_0 \in (0, \infty)$, as shown in *Red* in the right Figure. However, for $x_0 \in f_L^{-n}(0, \alpha d_1 + 1 - \alpha)$, as shown in *Green*, it is monotone increasing for the first *n* periods and monotone decreasing afterwards.



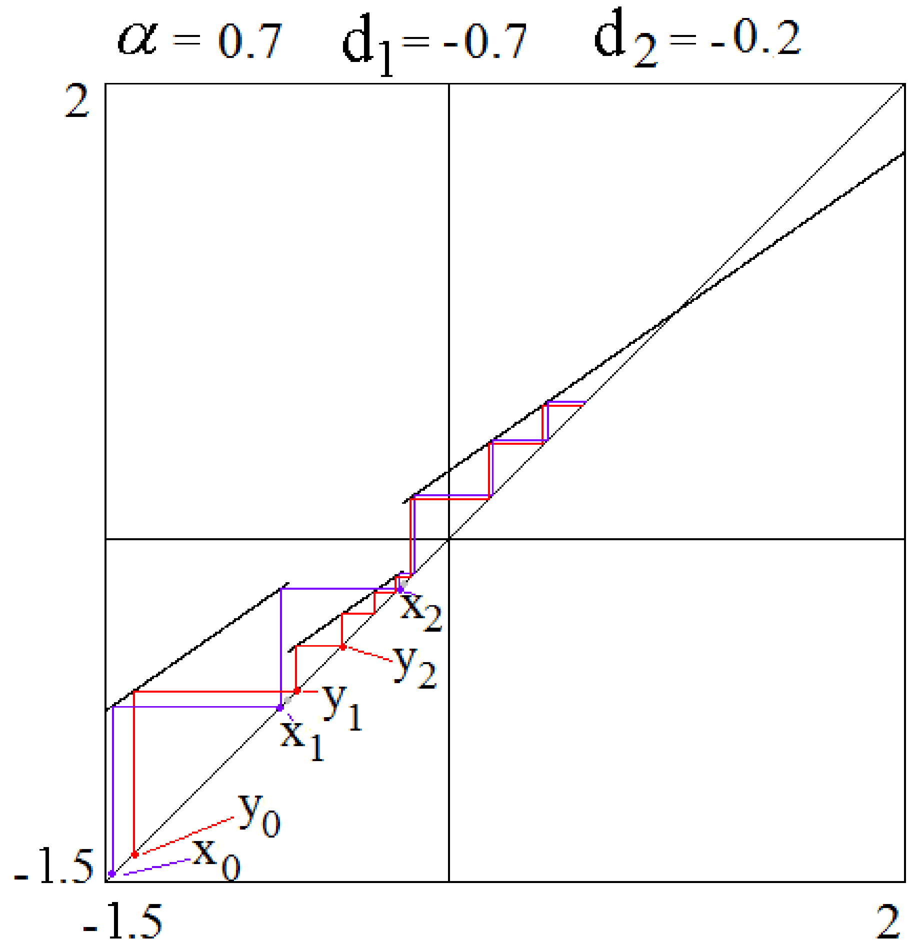


Case S-II ($d_1 < d_2 < 0$), Yellow; convergent to its unique steady state, $x_L^* = 1$.

- monotone decreasing for $x_0 \in (1, \infty)$.
- monotone increasing for $x_0 \in (-\infty, 1)$. However, it is possible to have **leapfrogging,** i.e.,

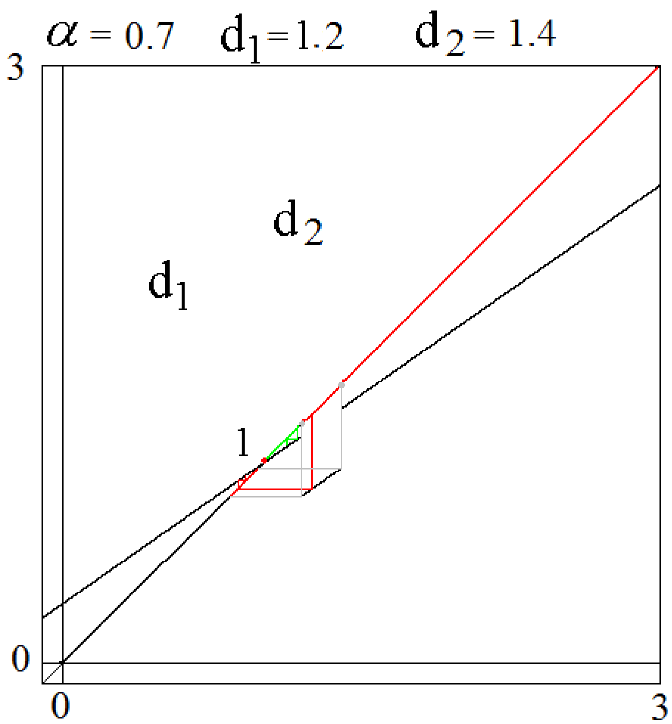
  $x_0 < y_0$ and then $x_t > y_t$ after some periods,

  as shown in this Figure.

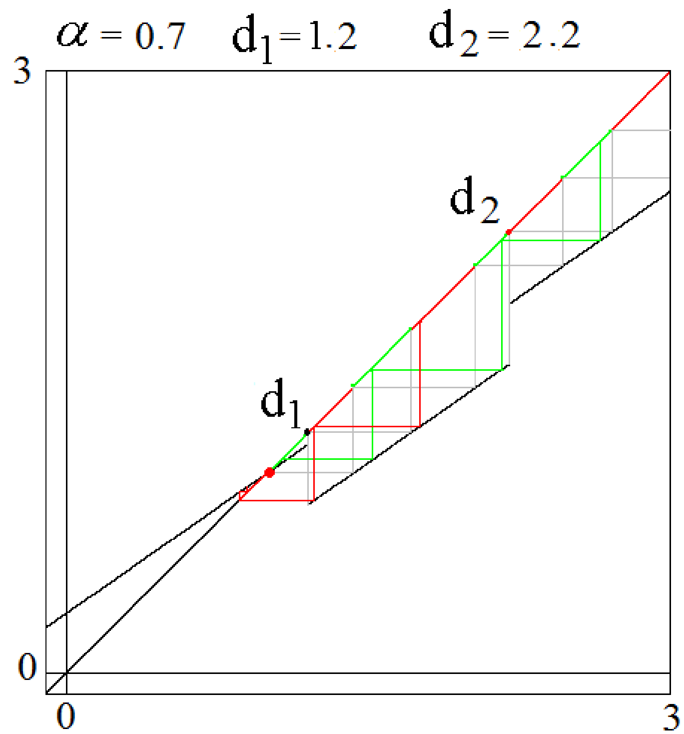


Case S-III $(1 < d_1 < d_2)$, Yellow; convergent to its unique steady state, $x_L^* = 1$.

- monotone increasing for $x_0 \in (-\infty, 1)$.
- monotone decreasing for the first $(n+1)$ periods and then monotone increasing afterwards for $x_0 \in f^{-n} \circ f_R^{-1}((0,1) \cap (\alpha d_1, \alpha d_2))$.
- monotone decreasing $x_0 \in (1, \infty) / \cup_{n>0} f^{-n} \circ f_R^{-1}((0,1) \cap (\alpha d_1, \alpha d_2))$.



$(\alpha d_1 < \alpha d_2 < 1)$



$(\alpha d_1 < 1 < \alpha d_2)$

Cases A: ***Orange*** ***and*** ***Yellow*** ***Stripes*** **(**$d_1 < 0 < d_2 < 1$): Both $x_R^* = 0$ and $x_L^* = 1$ are steady states

Case A-I; $(1-\alpha)+\alpha d_1 < d_2$ (Above the line *r*). **"Lower Steady State as a Trap"**

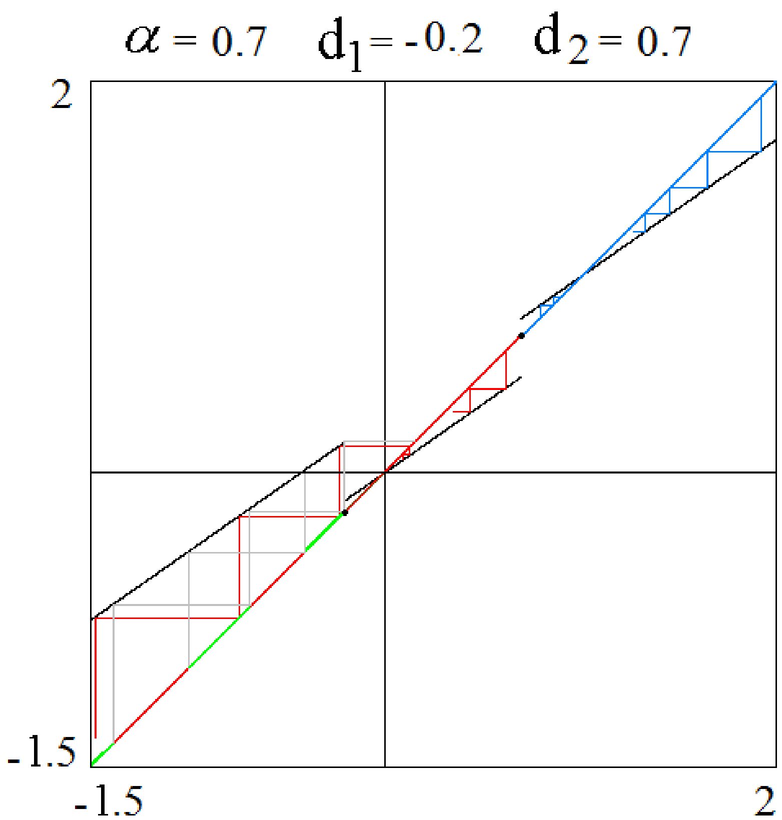
Two basins of attractions are simply connected and separated by d_2;

$B(0) = (-\infty, d_2)$ and $B(1) = [d_2, \infty)$.

- For $x_0 \in B(1)$, monotone decreasing for $x_0 > 1$ and monotone increasing for $d_2 < x_0 < 1$. *Blue*
- For $x_0 \in B(0)$,

➢ If $\alpha d_1 + 1 - \alpha < 0$, monotone increasing for $x_0 < 0$ & monotone decreasing for $0 < x_0 < d_2$.

➢ If $\alpha d_1 + 1 - \alpha > 0$, monotone increasing for $x_0 \in \cup_{n \geq 0} f_L^{-n}(d_1, 0)$ & monotone decreasing for $0 < x_0 < d_2$. *Red*. For $x_0 \in f_L^{-n}(0, \alpha d_1 + 1 - \alpha)$, monotone increasing for the first *n* periods and monotone decreasing afterwards. *Green*



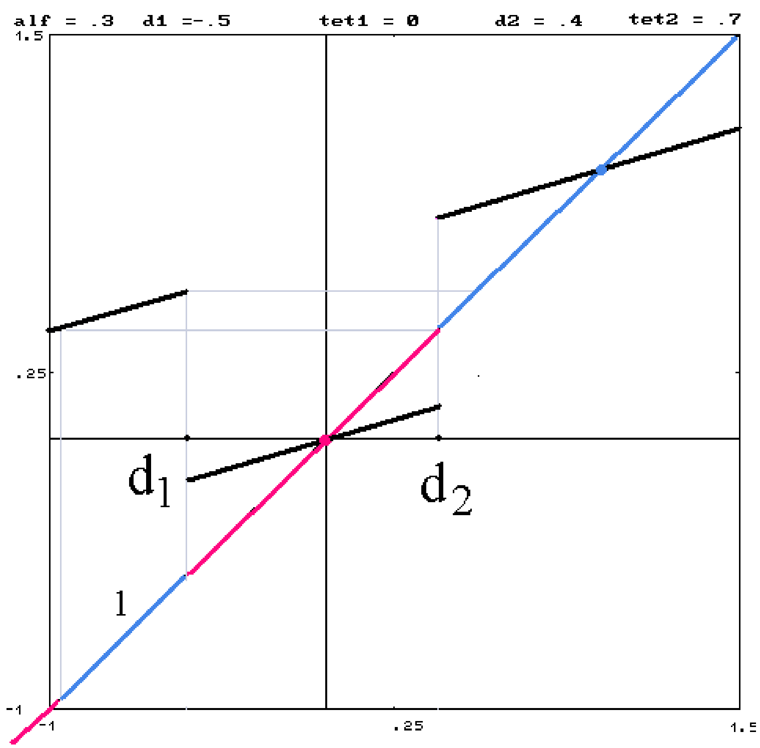
Case A-II: $(1-\alpha)+\alpha d_1 > d_2$, (Below the line r). **"Reversal of Fortune"**

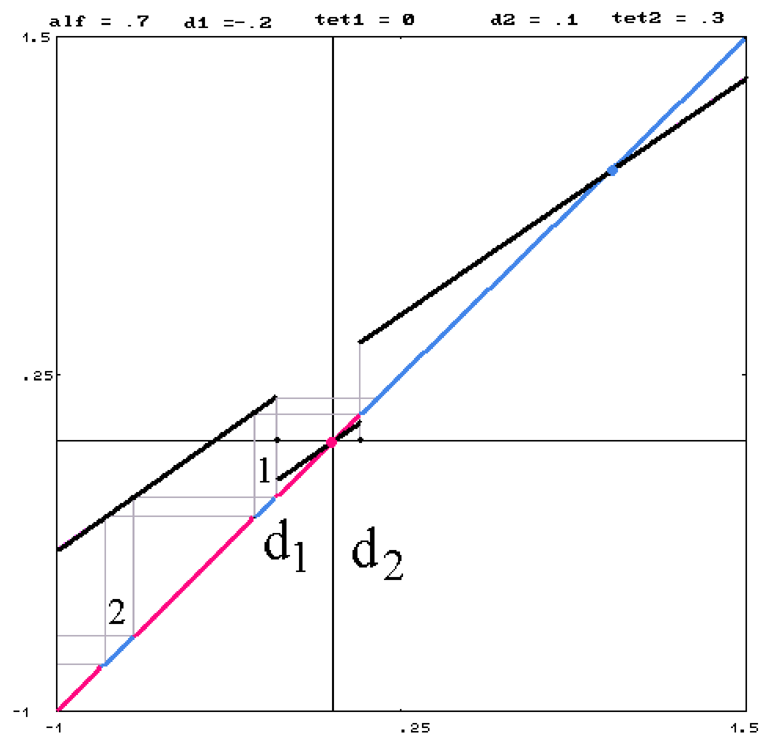
If $x_0 \in (d_1, d_2)$, $x_t \to x_R^* = 0$; If $x_0 \geq d_2$, $x_t \to x_L^* = 1$.

If $x_0 \leq d_1$, x_t converges to either $x_R^* = 0$ or $x_L^* = 1$.

Blue indicates $B(1)$, the basin of attraction for $x_L^* = 1$.

Red indicates $B(0)$, the basin of attraction for $x_R^* = 0$. Two basins of attraction, $B(1)$ & $B(0)$, are disconnected; they alternate and each accumulates to the origin in the space of $k = b^x$.





Case B: $(0 < d_1 < 1 < d_2)$; **"Cycles for all initial conditions."**
Neither $x_R^* = 0$ or $x_L^* = 1$ are steady states. For all x_0, the path enters $I = (\alpha d_1, 1 - \alpha + \alpha d_1]$ in a finite time and continues fluctuating inside *I*.

Figure shows the 2-cycle

$$x_0 = f_R \circ f_L(x_0);\ x_1 = f_L \circ f_R(x_1)$$

In symbolic dynamics (SD),

LR =*RL*,

It exists and is globally stable for:

$$d_1 \in (x_0, x_1) = \left(\frac{\alpha}{1+\alpha}, \frac{1}{1+\alpha}\right) \equiv \Pi_{LR},$$

shown in *Pink*.

Π_{LR} is symmetric around $d_1 = 0.5$.

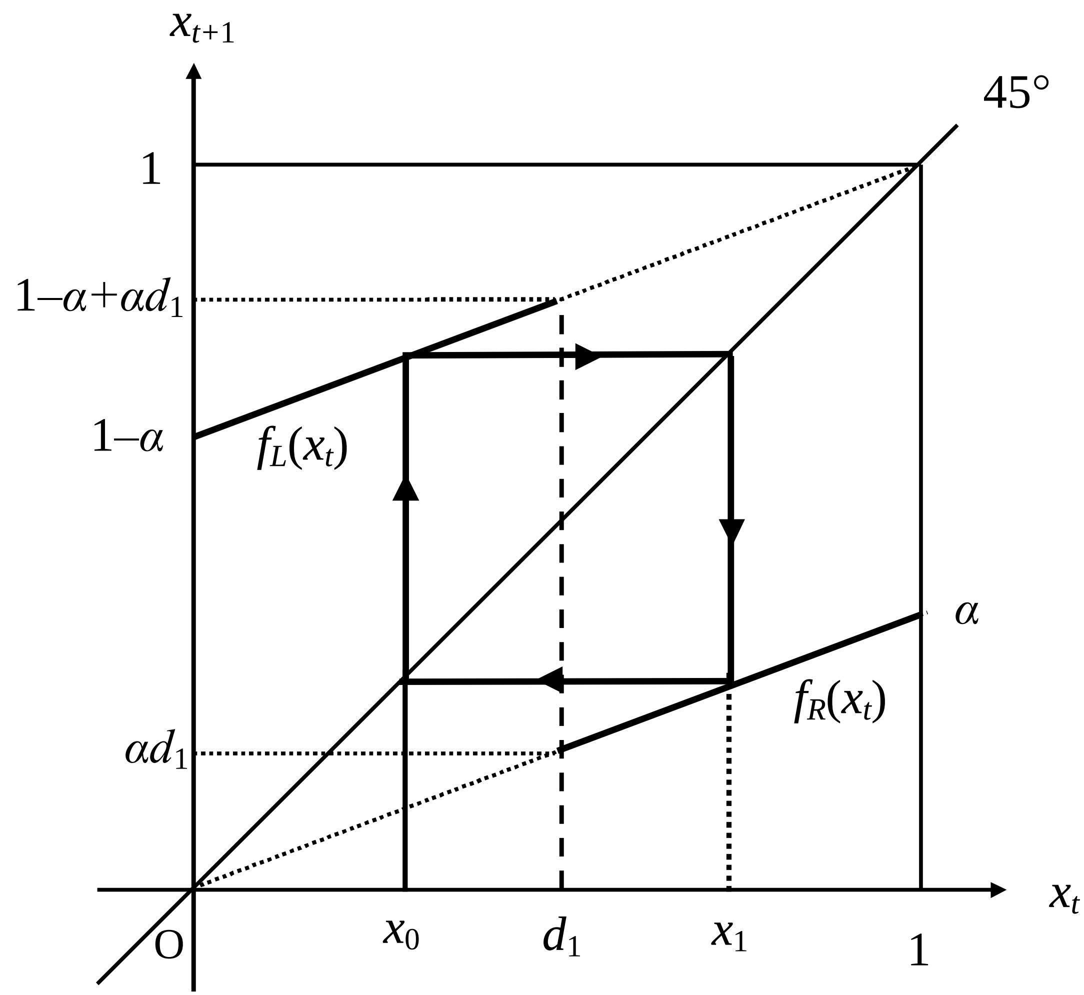


Figure shows the 3-cycle of the form, $x_0 < d_1 < x_2 < x_1$

$$x_0 = (f_R)^2 \circ f_L(x_0)\ ,\ x_1 = f_L \circ (f_R)^2(x_1)\ \&\ \ x_2 = f_R \circ f_L \circ f_R(x_2)$$

In symbolic dynamics (SD),

$$LR^2 = R^2L = RLR.$$

It exists and is globally stable for:

$$d_1 \in (x_0, x_2)$$

$$= \left(\frac{(1-\alpha)\alpha^2}{1-\alpha^3}, \frac{(1-\alpha)\alpha}{1-\alpha^3} \right) \equiv \Pi_{LR^2}$$

shown in *Red* on the left side of Π_{LR}.

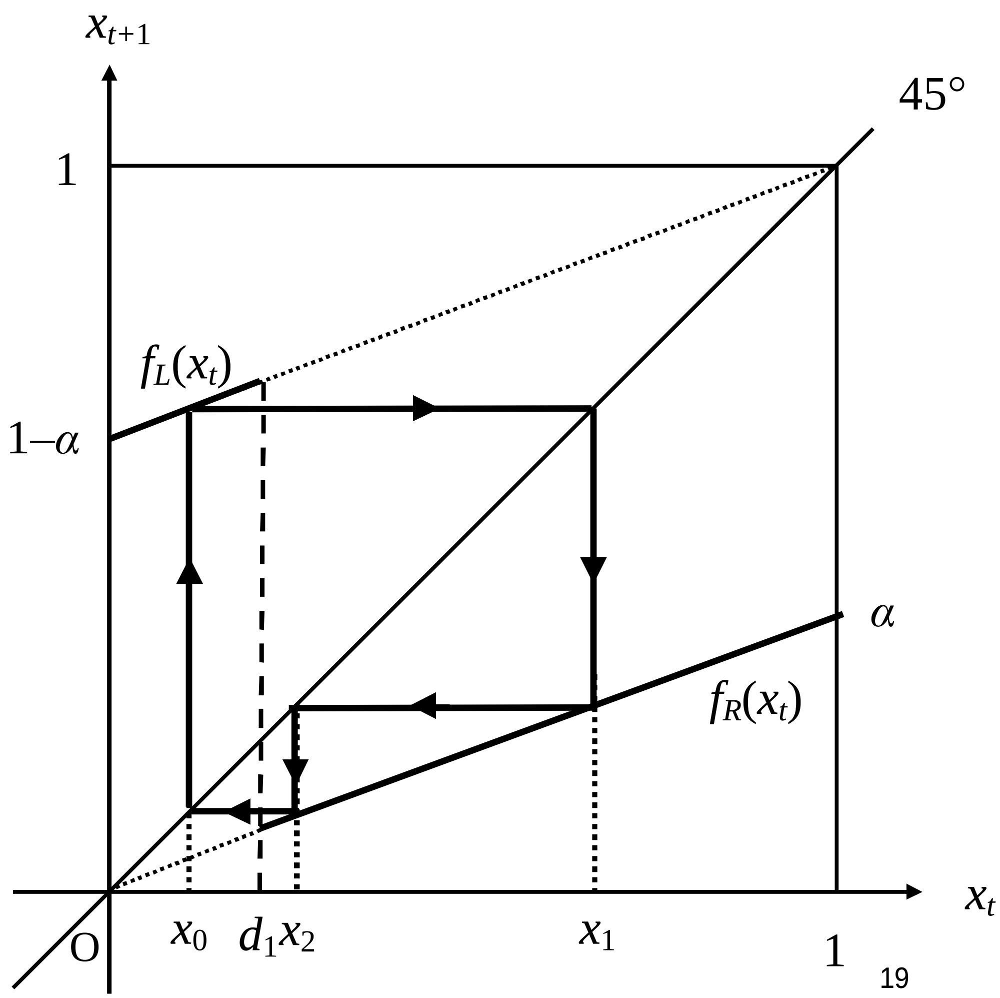


Figure shows the 3-cycle of the form, $x_1 < x_2 < d_1 < x_0$

$$x_0 = (f_L)^2 \circ f_R(x_0)\ ,\ x_1 = f_R \circ (f_L)^2(x_1)\ \&\ \ x_2 = f_L \circ f_R \circ f_L(x_2)$$

In symbolic dynamics,

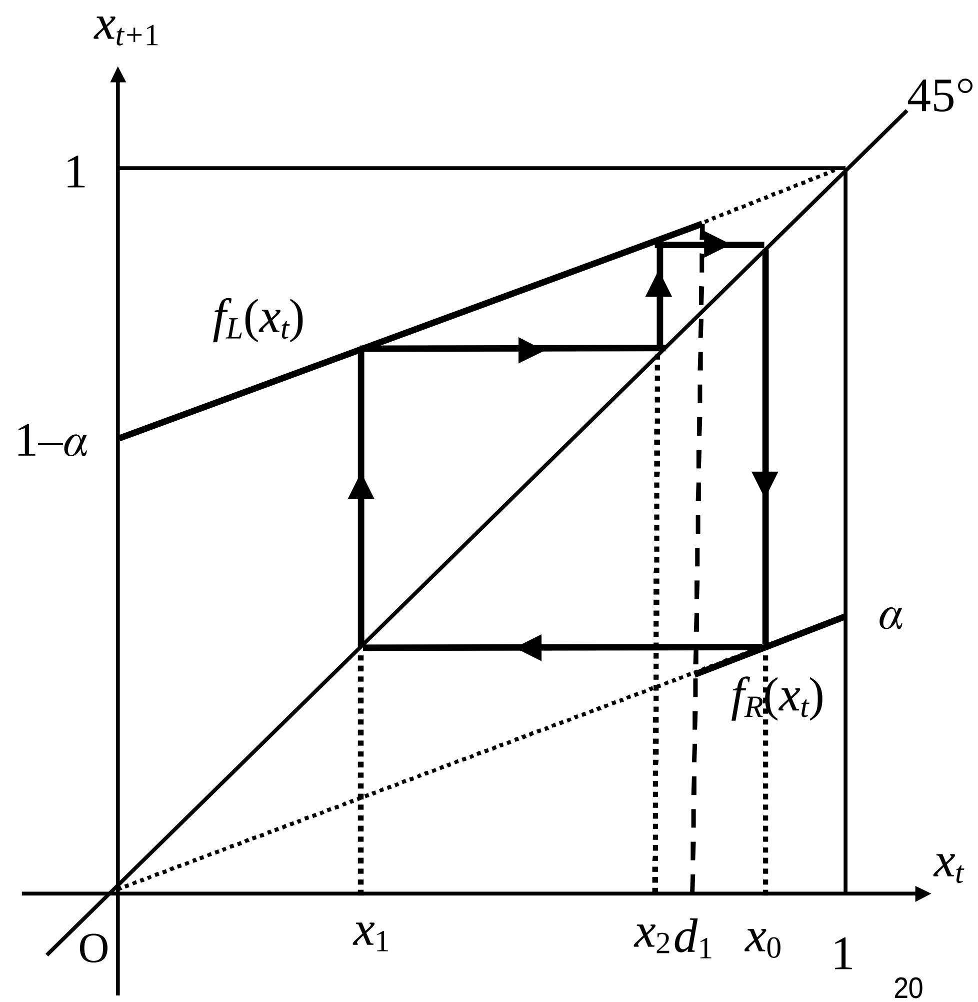
$$RL^2 = L^2R = LRL.$$

It exists and is globally stable for:

$$d_1 \in (x_2, x_0)$$
$$= \left(1 - \frac{(1-\alpha)\alpha^2}{1-\alpha^3}, 1 - \frac{(1-\alpha)\alpha}{1-\alpha^3}\right) \equiv \Pi_{RL^2},$$

shown in *Red* on the right side of Π_{LR}.

Π_{LR^2} & Π_{RL^2} are symmetric around $d_1 = 0.5$.



More generally,

- The (n+1)-cycle of the form, $x_0 < d_1 < x_n < ... < x_1$; $LR^n = \ldots = RLR^{n-1}$ in SD

$$x_0 = (f_R)^n \circ f_L(x_0)\ , ...,\ \ x_n = (f_R)^{n-1} \circ f_L \circ f_R(x_n)\ ;$$

exists and is globally stable if $d_1 \in (x_0, x_n) = \left(\frac{(1-\alpha)\alpha^n}{1-\alpha^{n+1}}, \frac{(1-\alpha)\alpha^{n-1}}{1-\alpha^{n+1}} \right) \equiv \Pi_{LR^n}$.

- The (n+1)-cycle of the form, $x_1 < ... < x_n < d_1 < x_0$; $RL^n = \ldots = LRL^{n-1}$ in SD

$$x_0 = (f_L)^n \circ f_R(x_0)\ , ...,\ \ x_n = (f_L)^{n-1} \circ f_R \circ f_L(x_n)\ ;$$

exists and is globally stable if $d_1 \in (x_n, x_0) = \left(1 - \frac{(1-\alpha)\alpha^n}{1-\alpha^{n+1}}, 1 - \frac{(1-\alpha)\alpha^{n-1}}{1-\alpha^{n+1}} \right) \equiv \Pi_{RL^n}$.

The periodicity regions, Π_{LR^n} accumulates to $d_1 = 0$
The periodicity regions, Π_{RL^n} accumulates to $d_1 = 1$.
They are symmetric around $d_1 = 0.5$.

Cycles of the Higher Levels of Complexity

Cycles of the form, RL^n and LR^n for $n \geq 1$ are called the **First Level of Complexity**.

In the gap between $\Pi_{LR^{n+1}}$ and Π_{LR^n}, i.e, $d_1 \in \left(\frac{(1-\alpha)\alpha^n}{1-\alpha^{n+2}}, \frac{(1-\alpha)\alpha^n}{1-\alpha^{n+1}} \right)$ for any integer $n \geq 1$, there exist two infinite sequences of periodicity regions of cycles of the **Second Level of Complexity**,

$$\Pi_{LR^n(LR^{n+1})^m} \quad \text{and} \quad \Pi_{LR^{n+1}(LR^n)^m} \quad \text{for each integer } m \geq 1.$$

accumulating to $\Pi_{LR^{n+1}}$ and Π_{LR^n}, respectively.

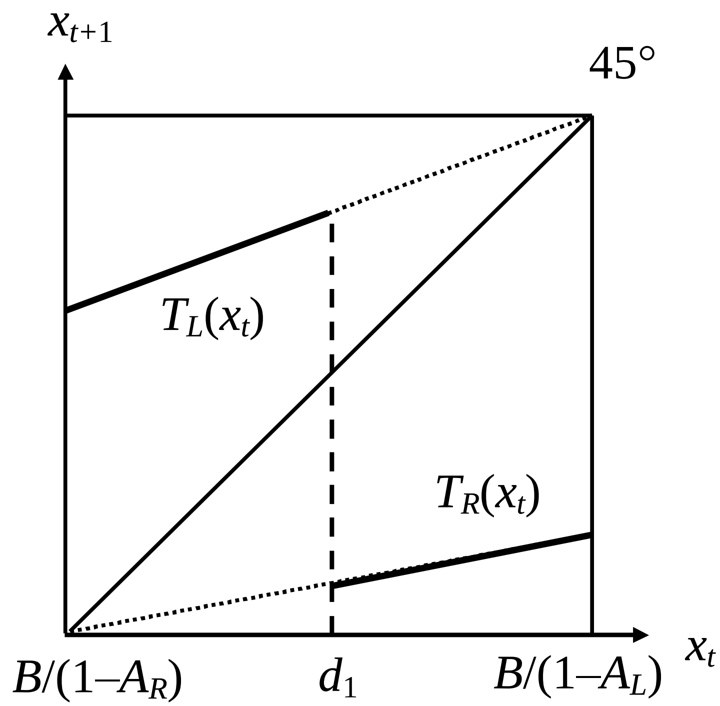
To see this, define a new map on the interval (the gap between $\Pi_{LR^{n+1}}$ and Π_{LR^n}), as follows:

$$x_{t+1} = \begin{cases} T_L(x_t) \equiv f_R^n \circ f_L(x_t) & \text{if } \dfrac{(1-\alpha)\alpha^n}{1-\alpha^{n+2}} < x_t < d_1 \\ T_R(x_t) \equiv f_R^n \circ f_L \circ f_R(x_t) & \text{if } d_1 < x_t < \dfrac{(1-\alpha)\alpha^n}{1-\alpha^{n+1}}, \end{cases}$$

which can be rewritten as:

$$x_{t+1} = \begin{cases} T_L(x_t) \equiv A_L x_t + B & \text{if } \dfrac{B}{1-A_R} < x_t < d_1 \\ T_R(x_t) \equiv A_R x_t + B & \text{if } d_1 < x_t < \dfrac{B}{1-A_L}, \end{cases}$$

where $A_L = \alpha^{n+1} > A_R = \alpha^{n+2}$ and $B = (1-\alpha)\alpha^n$.



Therefore, following the same procedure, we can find:

- The (m+1)-cycle of the symbolic sequence, $T_L(T_R)^m$

  - ➢ The [n+1+m(n+2)]-cycle of f with the symbolic dynamics $LR^n(RLR^n)^m$,
  - ➢ Its periodicity region:

$$d_1 \in \left(\frac{B(1-A_R^{m+1})}{(1-A_R)(1-A_L A_R^m)}, \frac{B[(1-A_R^m)+A_L(1-A_R)A_R^{m-1}]}{(1-A_R)(1-A_L A_R^m)} \right)$$

accumulates to the right edge of $\Pi_{LR^{n+1}}$, as $m \to \infty$.

- The (m+1)-cycle of the symbolic sequence, $T_R(T_L)^m$

  - ➢ The [n+2+m(n+1)]-cycle of f with the symbolic dynamics, $RLR^n(LR^n)^m$,
  - ➢ Its periodicity region:

$$d_1 \in \left(\frac{B[(1-A_L^m)+A_R(1-A_L)A_L^{m-1}]}{(1-A_L)(1-A_R A_L^m)}, \frac{B(1-A_L^{m+1})}{(1-A_L)(1-A_R A_L^m)} \right)$$

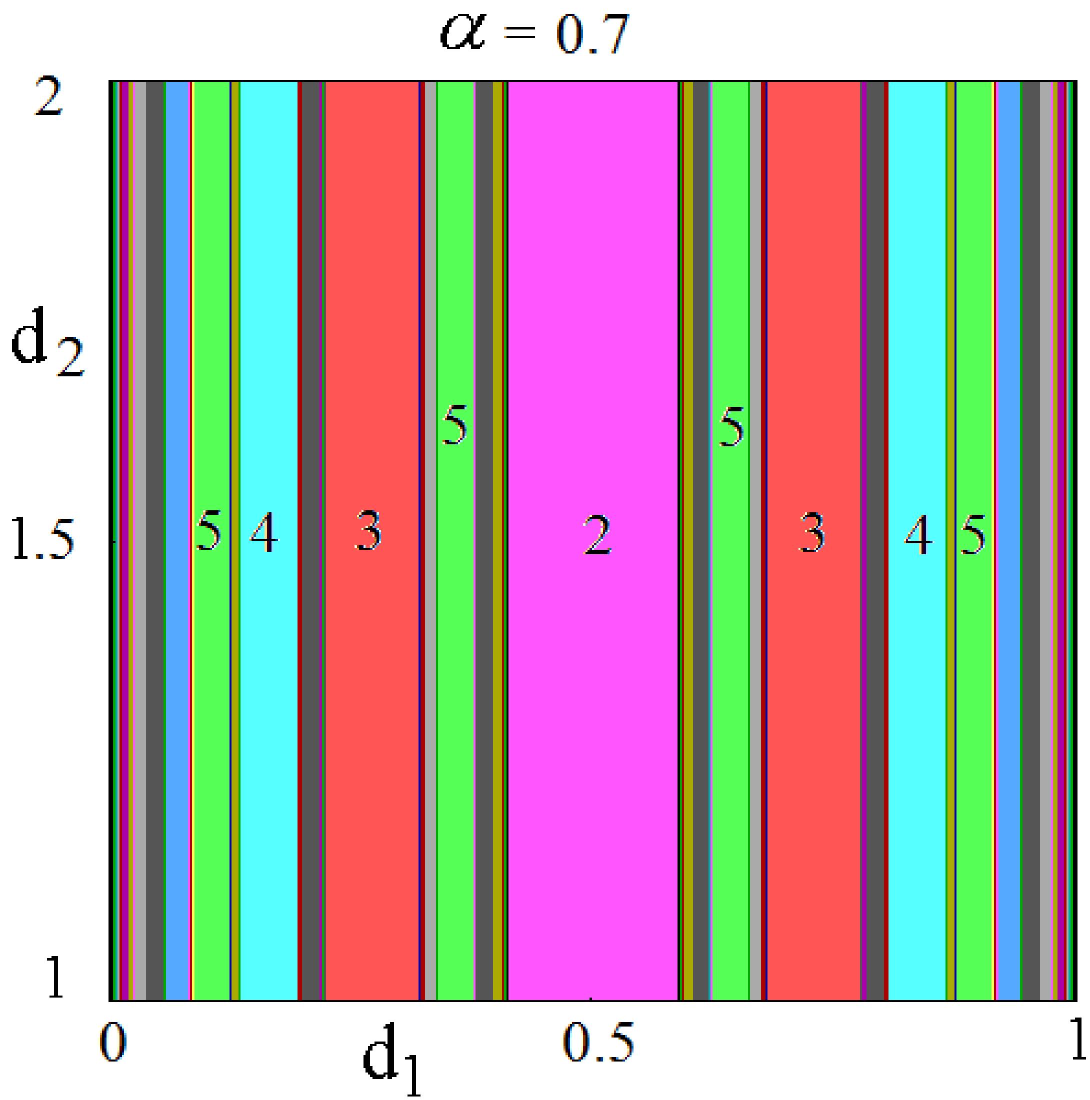
accumulates to the left edge of Π_{LR^n}, as $m \to \infty$.

- This procedure can be repeated infinitely many times. Thus, between the periodicity regions of the cycles of the kth-level of complexity, there are two infinite sequences of the periodicity regions of the cycles of the (k+1)th-level of complexity.

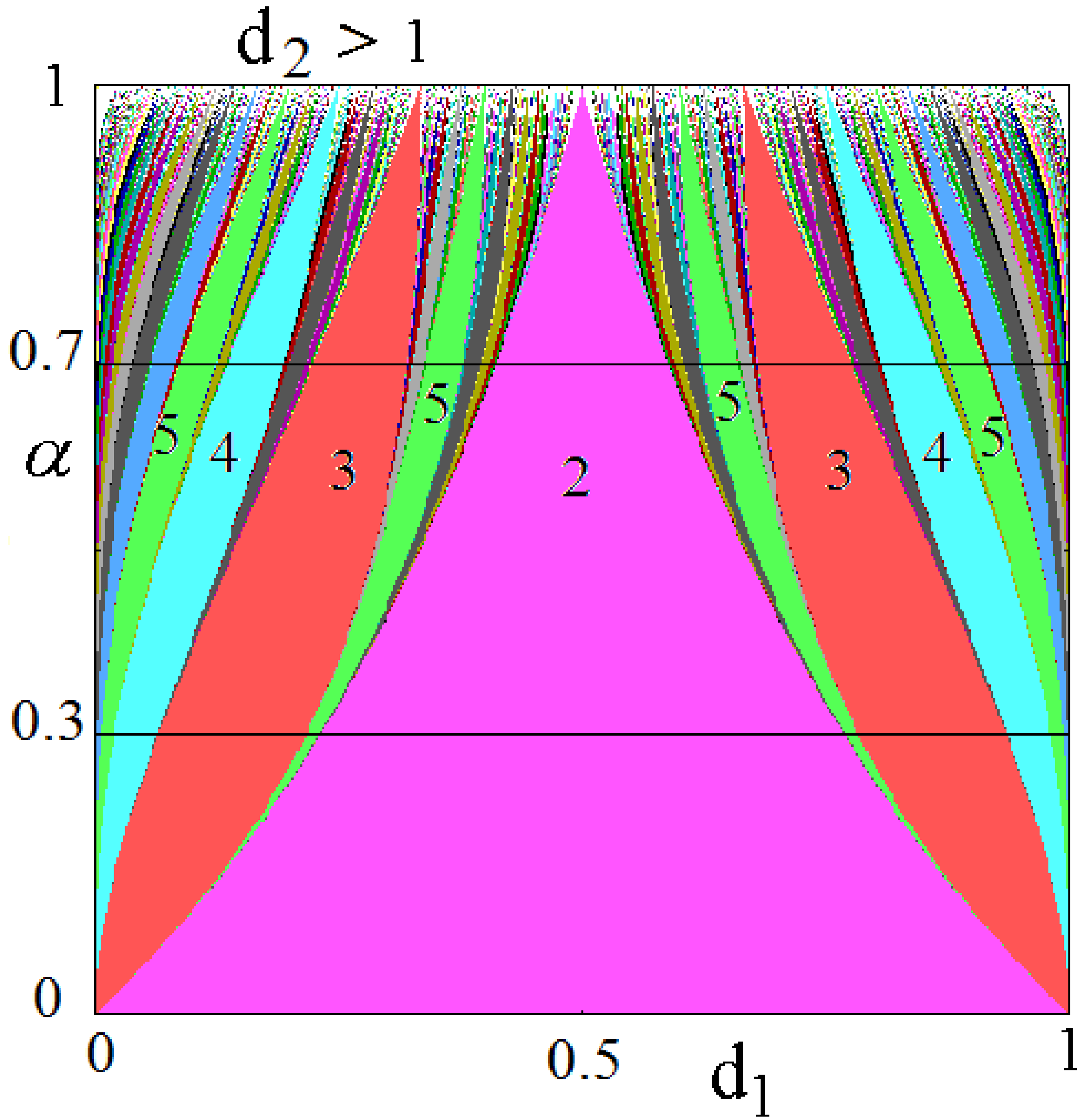
- The union of all the periodicity regions thus constructed does not cover the entire interval of $d_1 \in (0,1)$.

- The set of d_1 left is a set of measure zero. On this set, the trajectory is quasi-periodic, dense in the invariant set, which is a Cantor set.

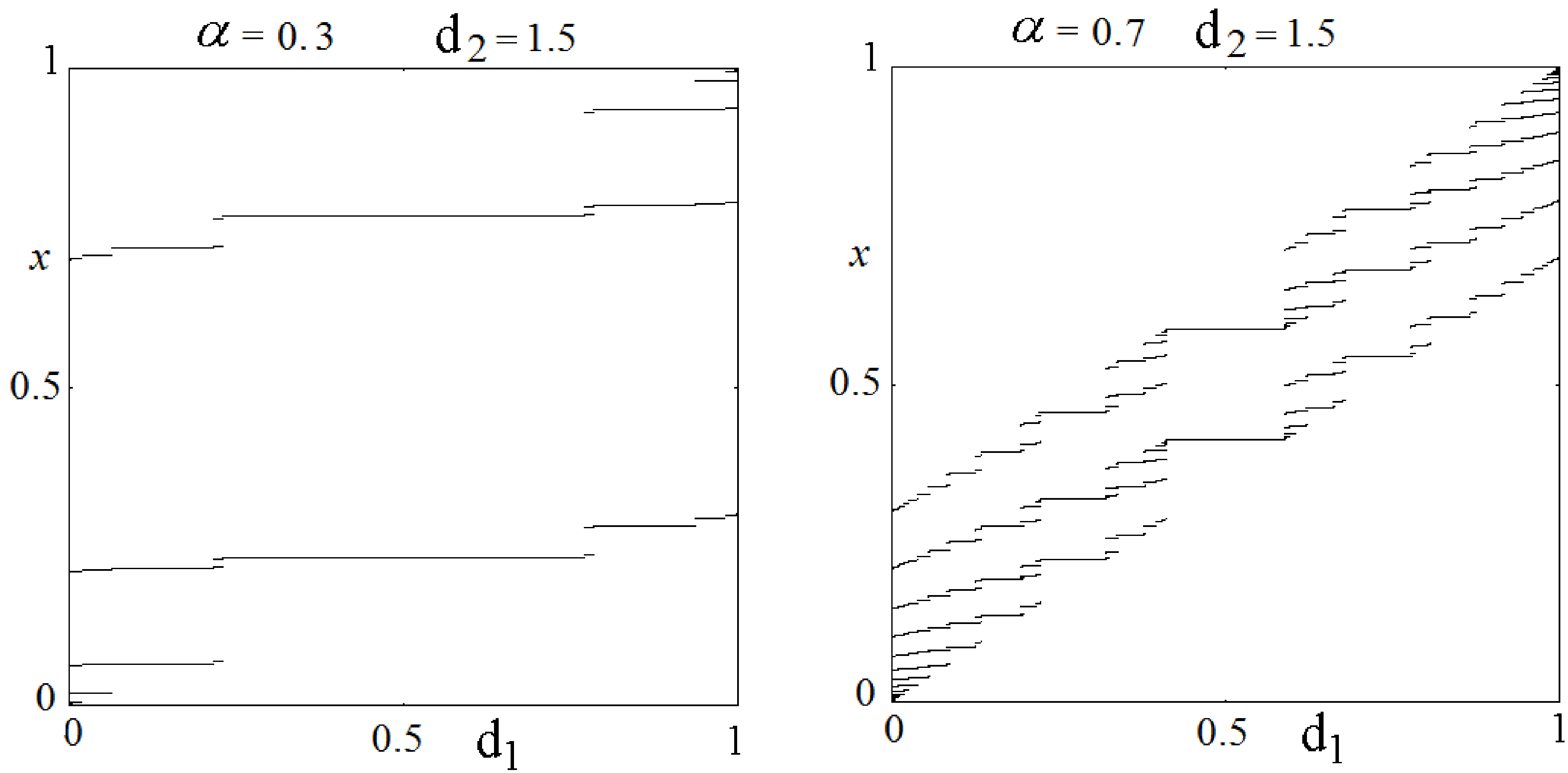
This Figure shows the periodicity regions for Case B ($\alpha = 0.7$).



This Figure shows how the periodicity regions for Case B change with α.



Bifurcation Diagram, tracing the orbit of stable cycles as a function of $d_1 \in (0,1)$



The Rotation (Winding) Number:

We can calculate, along the stable cycles, what fraction of the periods the economy is in an expansionary stage (that is, on the left side of d_1).

For the *k*-period cycles, along which the periodic orbit visits *p* times on the *L* side and $k - p$ times, we can associate **its rotation number**, *p/k*. For example,

On cycles of first level of complexity:

$$\omega = \frac{1}{1+n} \text{ for } LR^{n} \qquad \& \qquad \omega = \frac{n}{1+n} \text{ for } RL^{n}.$$

On cycles of second level of complexity between LR^{n+1} and LR^{n}:

$$\omega = \frac{1+m}{(1+n)+m(2+n)} \text{ for } LR^{n}(RLR^{n})^{m} \qquad \& \quad \omega = \frac{1+m}{(2+n)+m(1+n)} \text{ with } RLR^{n}(LR^{n})^{m},$$

and so on.

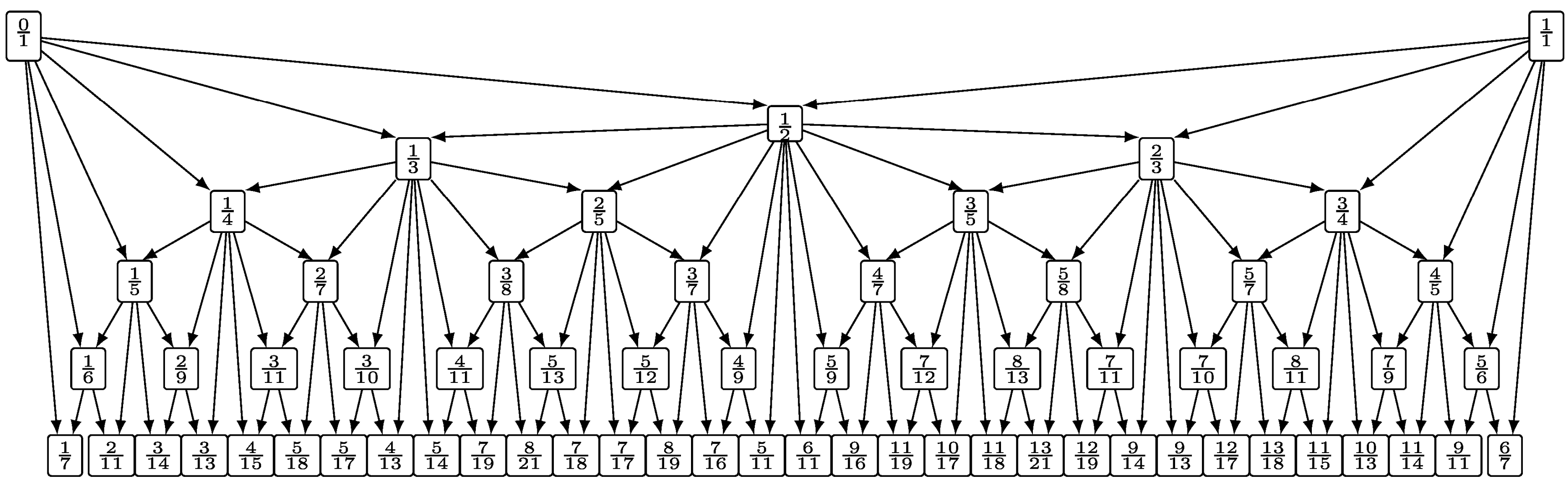
More generally,

Between the two periodicity regions of cycles whose rotation numbers, $p_1/k_1 < p_2/k_2$, are **Farey neighhors**, (i.e., they satisfy $\left|p_1k_2 - p_2k_1\right| = 1$), we can find the periodicity regions of cycles with the rotation number:

Farey composition rule: $$\frac{p_1}{k_1} \oplus \frac{p_2}{k_2} \equiv \frac{p_1 + p_2}{k_1 + k_2}\ .$$

Since $\frac{p_1}{k_1} < \frac{p_1}{k_1} \oplus \frac{p_2}{k_2} < \frac{p_2}{k_2}$ and $\frac{p_1}{k_1} \oplus \frac{p_2}{k_2}$ is a Farey neighbor of both $\frac{p_1}{k_1}$ and $\frac{p_2}{k_2}$,

this can repeat itself *ad infinitum*. Thus, the periodicity region for the rotation number equal to any rational number between 0 and 1 can be found, as shown by **Farey tree**.



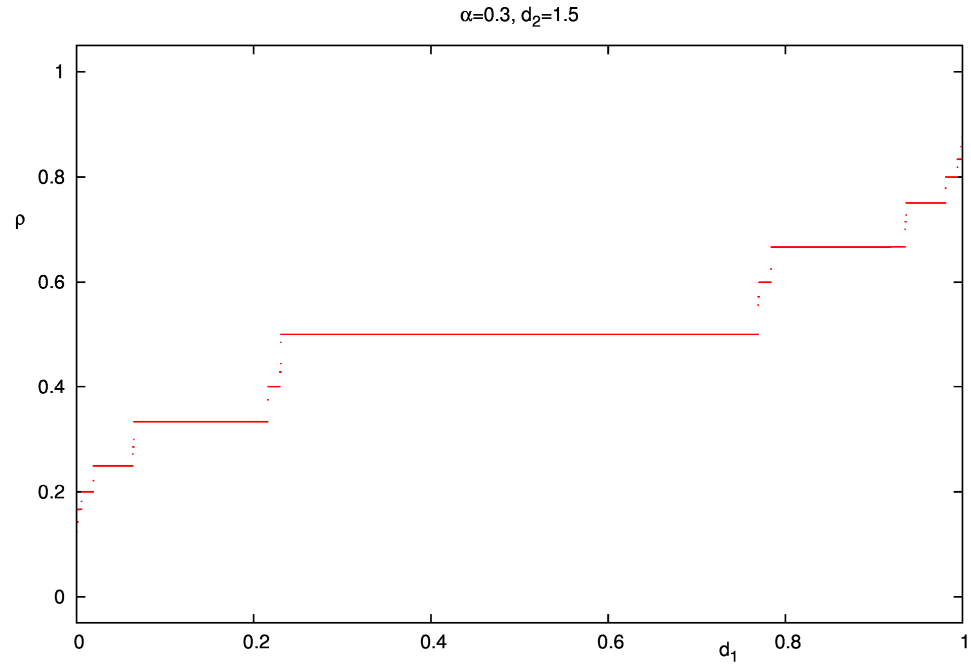
Furthermore,

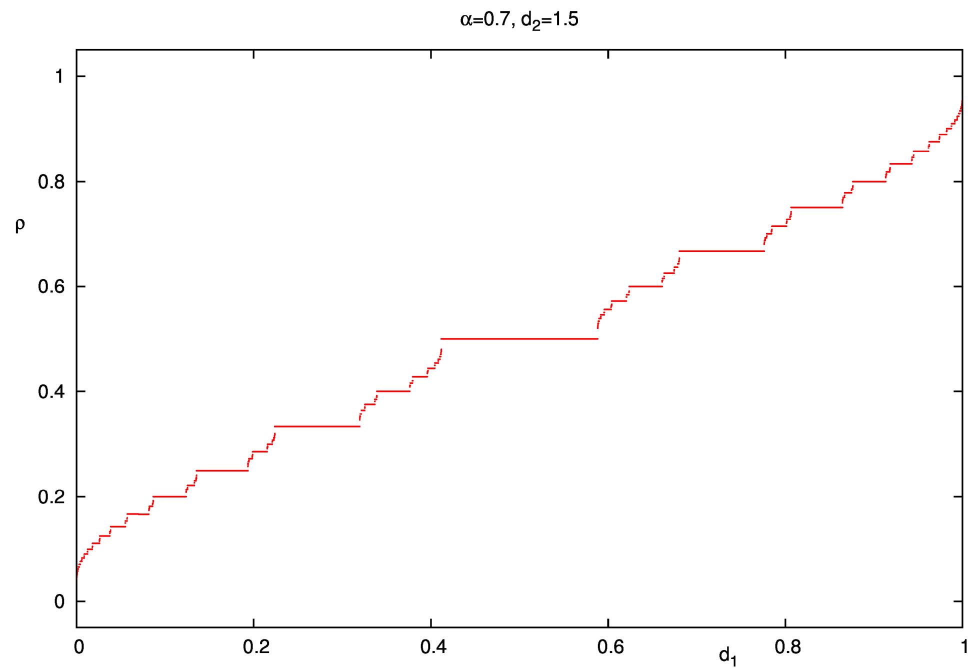
The rotation number can be expressed as a function of d_1, $\omega(d_1)$.

It is
- continuous;
- non-decreasing;
- goes up from zero to one.

Yet,
- have zero derivate almost everywhere. That is, it is not absolutely continuous.

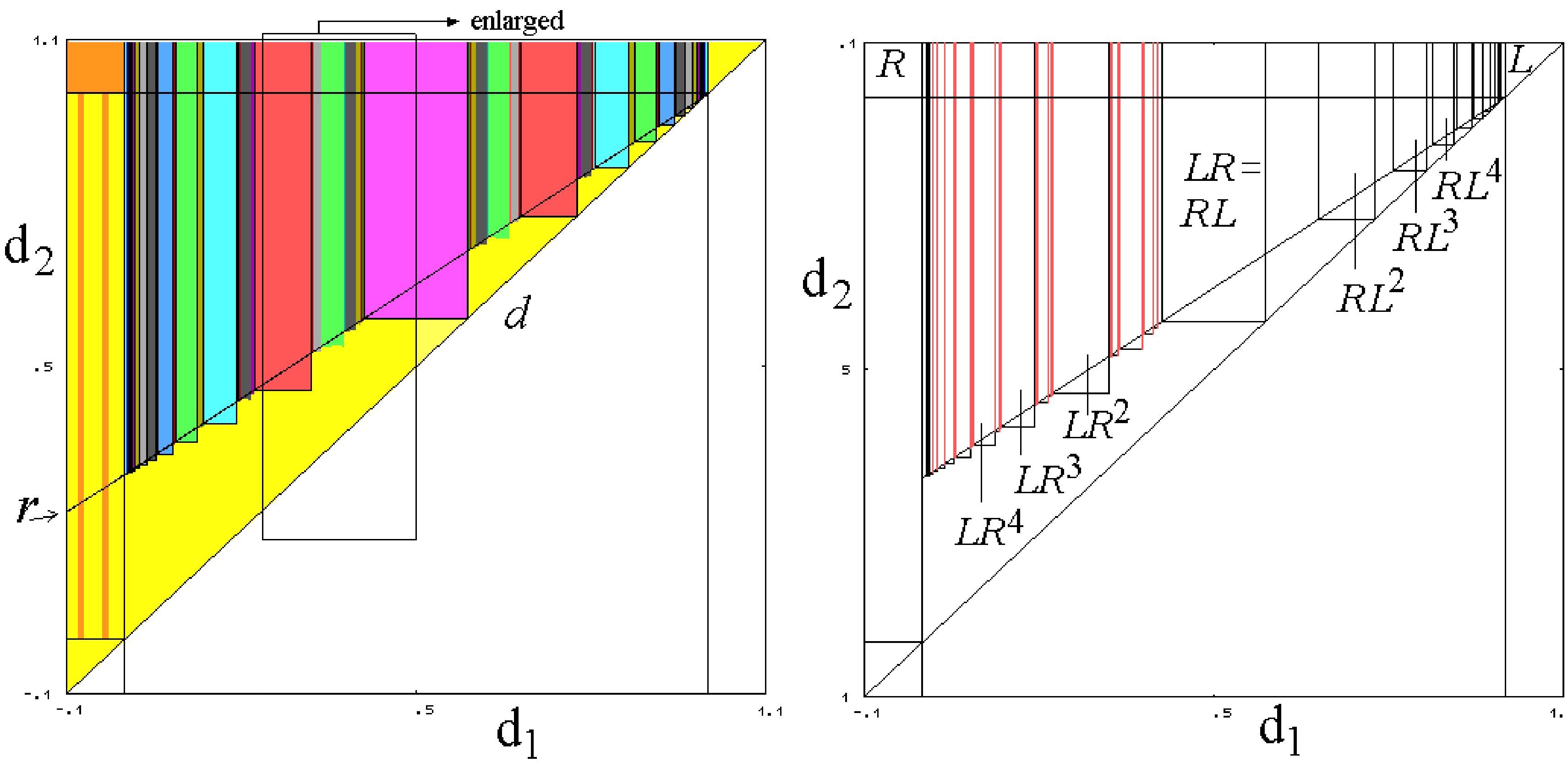
A **singular (Cantor) function,** often referred to as **the Devil's staircase**.





Cases C: $(0 < d_1 < d_2 < 1)$; $x_L^* = 1$ is the unique steady state. Furthermore,

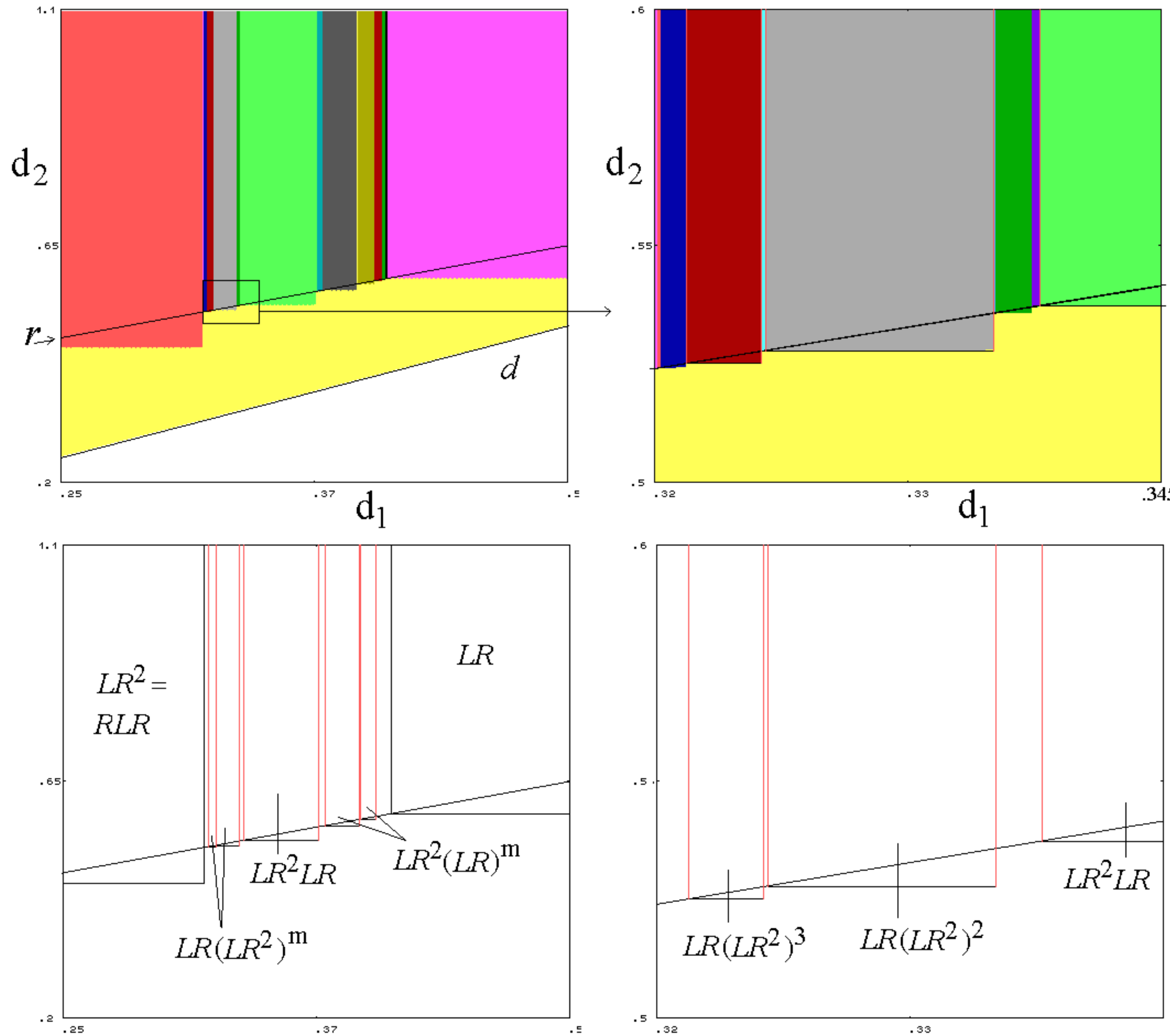
- All the stable cycles discussed in Case B survive as long as d_2 is greater than the rightmost location along its orbit.
- As soon as d_2 collides with the orbit, the stable cycles are destroyed.
- This explains the lower boundary of the periodicity regions, as shown in the Figure.



In this Figure,

Between the periodicity region of *LR* and LR^2,

A few regions of the cycles of the second level of complexity, can be seen.



Case C-I: $(0 < d_1 < d_2 < 1 \ \& \ 1 - \alpha + \alpha d_1 < d_2$; Above the line, "r").
"Cycles as a Poverty Trap."

All the stable cycles survive and co-exist with the steady state, $x_L^* = 1$.

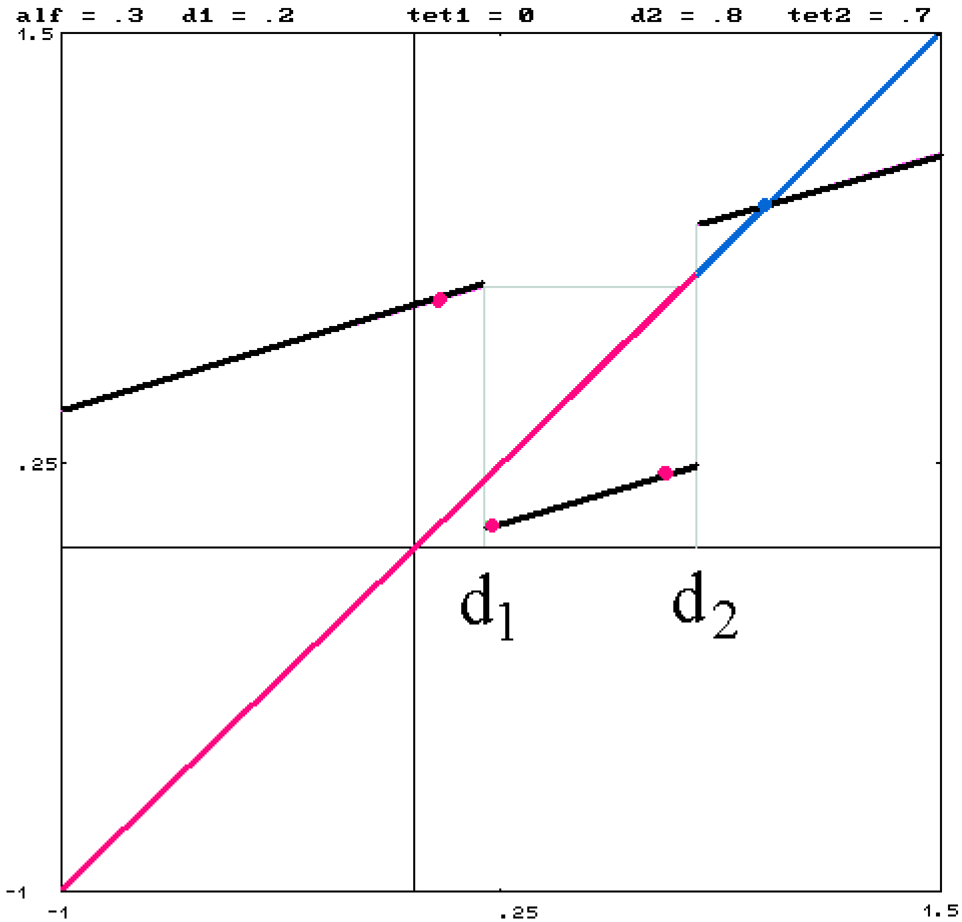
Furthermore, two basins of attraction are simply connected, separated at d_2.

Blue: The basin of attraction for $x_L^* = 1$.

$$B(1) = [d_2, \infty)$$

Red: The basin of attraction for the cycles:

$$B(C) = (-\infty, d_2)$$



Case C-II: $(0 < d_1 < d_2 < 1 \ \& \ 1-\alpha+\alpha d_1 > d_2 > d_1)$. **"Growth miracle"**
If $x_0 \geq d_2$, $x_t \to x_L^* = 1$.
If $x_0 < d_2$,

Case C-IIa: For some initial conditions, $x_0 < d_2$, the path eventually crosses over d_2 and converges to $x_L^* = 1$. For other initial conditions, the path fluctuate forever inside $I = (\alpha d_1, d_2]$. The periodicity regions are shown in the figure, the area below the line, r.

$B^0(1) \equiv (d_2, +\infty)$; The immediate basin of attraction for $x_L^* = 1$.

$B^n(1) \equiv f^{-n}((d_2, \alpha d_1 + (1-\alpha)))$, The set of initial conditions from which, after n iterations, the path escape to the immediate basin of attraction for $x_L^* = 1$.

$B(1) \equiv \cup_{n \geq 0} B^n(1)$: The basin of attraction for $x_L^* = 1$.

$B(C) \equiv R \setminus Cl(B(1))$; The basin of attraction for the cycles.

Again, two basins of attraction are disconnected.

Case C-IIb: (*Yellow*) For all $x_0 < d_2$, the path eventually crosses over d_2 so that $x_L^* = 1$ is globally attracting. That is,

$$B(1) \equiv \cup_{n \geq 0} B^n(1) = R$$

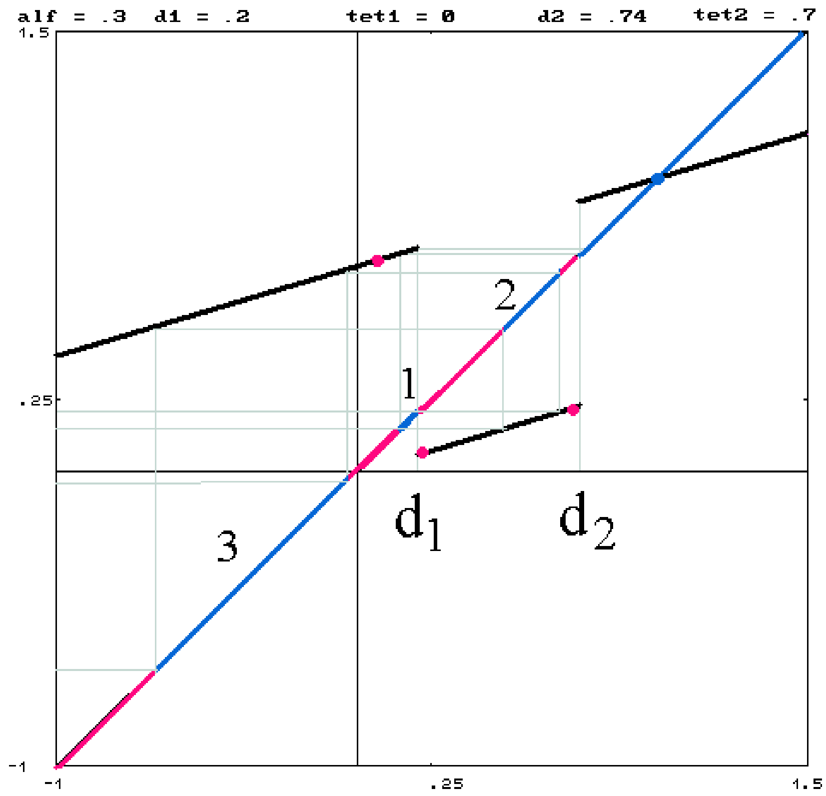
However,

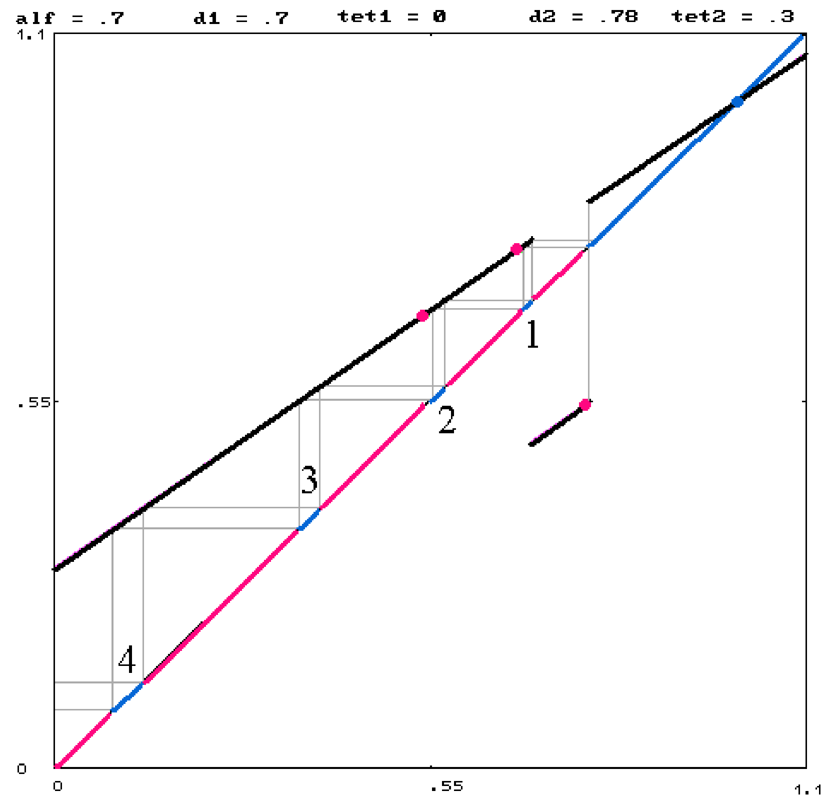
- The equilibrium trajectory changes its direction many times (unlike the other area of yellow, where it changes its direction at most once).

Furthermore,

- The structure of $B^n(1) \equiv f^{-n}((d_2, \alpha d_1 + (1-\alpha)))$ can be quite complicated.

This figure shows the co-existence of period-7 cycles and the intervals from which the orbit will eventually escape.



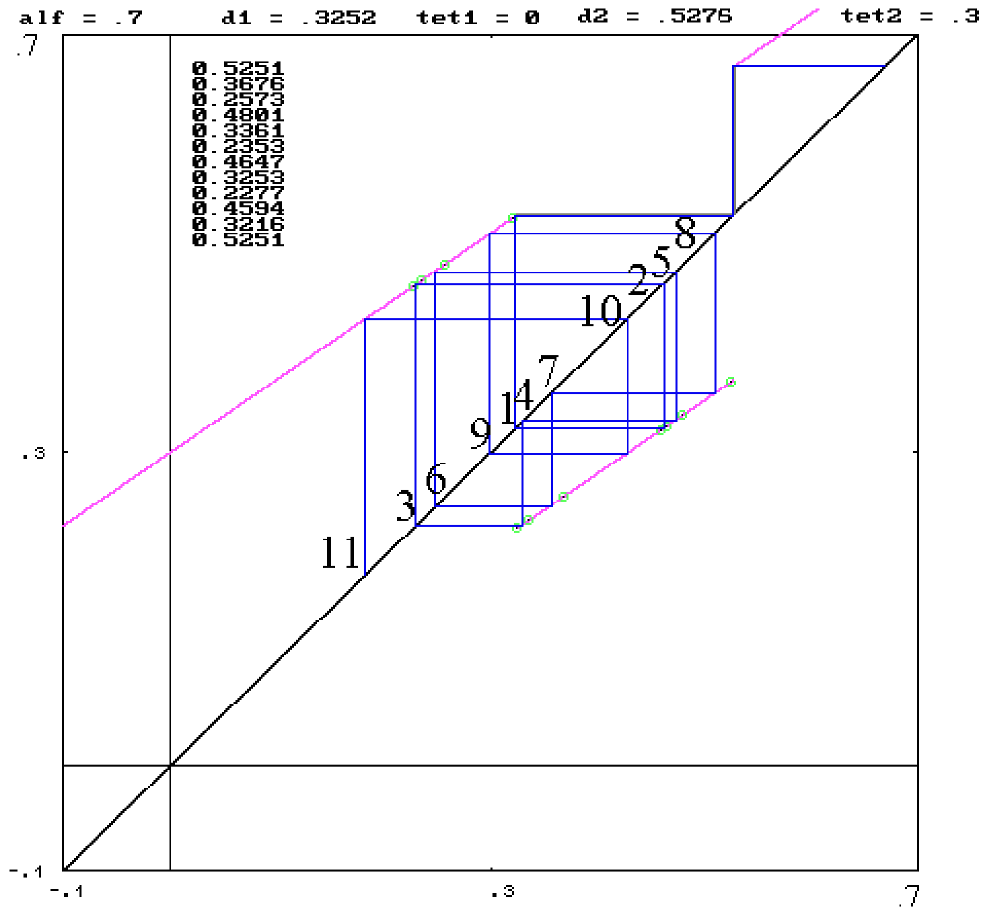


In this figure, $\alpha = 0.7$ and $d_1 = 0.3252$ and $\alpha d_1 + (1-\alpha) = 0.52764 > d_2 = 0.5276$.

The 11-cycle exists.

Furthermore, some paths escape above d_2, as shown.

The numbers of iteration required before the escape are indicated.



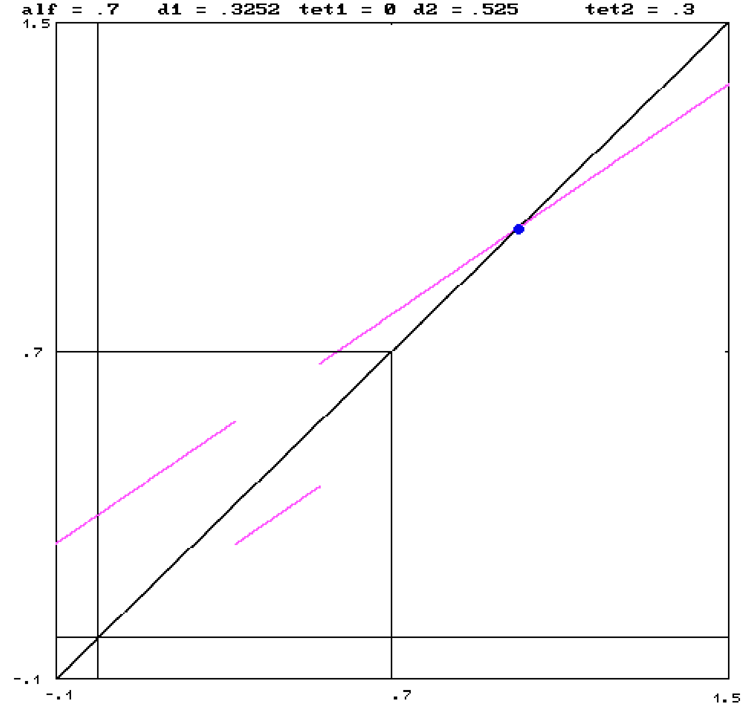
.52761	.52762	.52763
.3251571428571429	.3251714285714285	.3251857142857143
.4645102040816327	.4645306122448979	.4645510204081633
.2350145772594752	.2350437317784255	.2350728862973761
.3357351103706789	.335776759683465	.3358184089962516
.4796215862438271	.479681085262093	.4797405842803595
.2566022660626101	.2566872646601328	.2567722632576565
.3665746658037287	.3666960923716183	.3668175189395093
.5236780940053267	.5238515605308832	.5240250270564418
.3195401342933238	.3197879436155474	.320035752937774
.4564859061333198	.4568399194507821	.4571939327682485
.2235512944761711	.2240570277868315	.2245627610974979

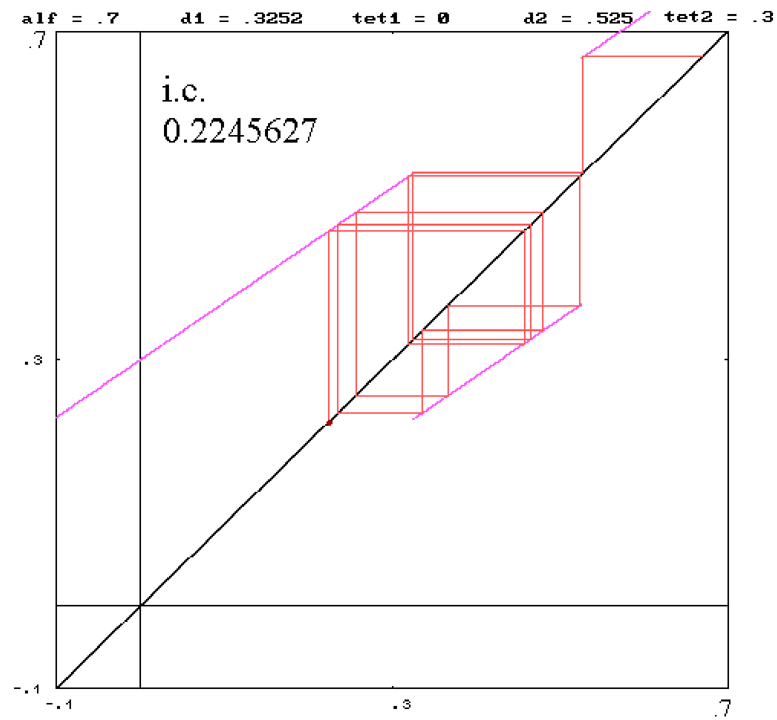
Then, at $d_2 = 0.525$,

The 11-cycle no longer exist.

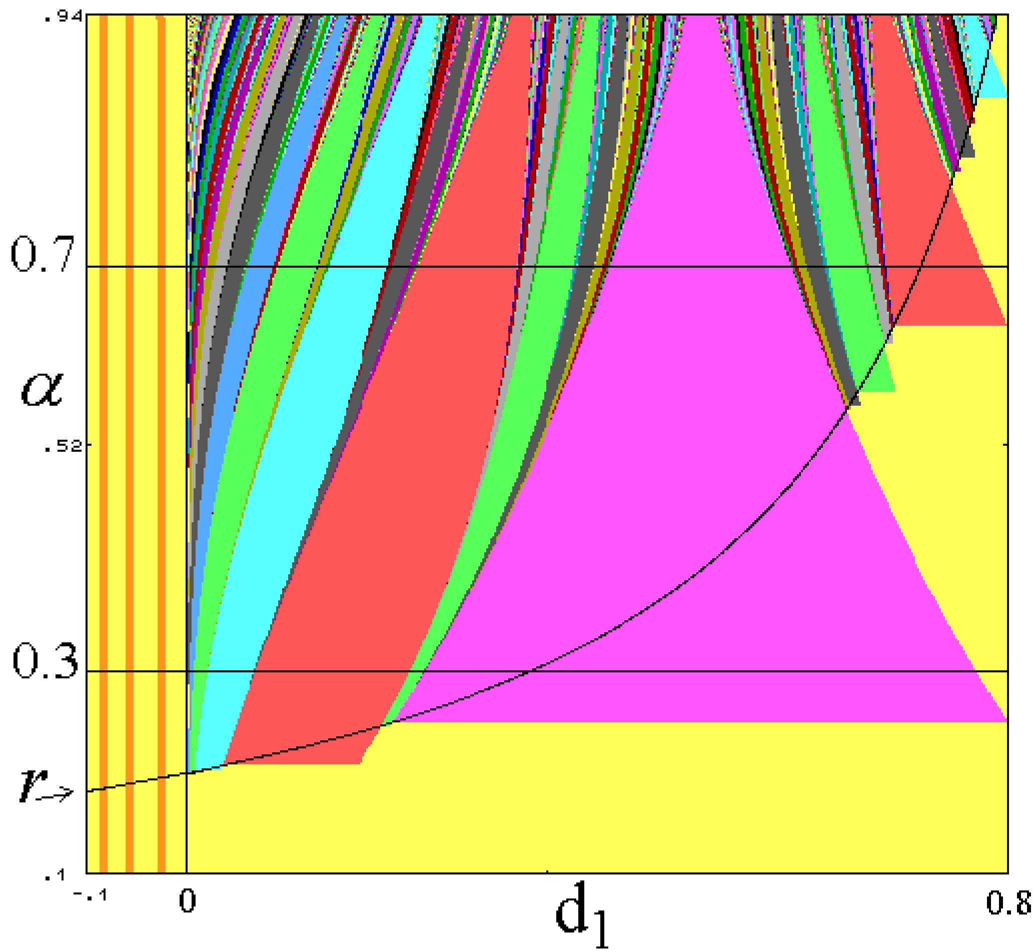
All converges to $x_L^* = 1$.

Yet, some paths fluctuate for long time before the escape.

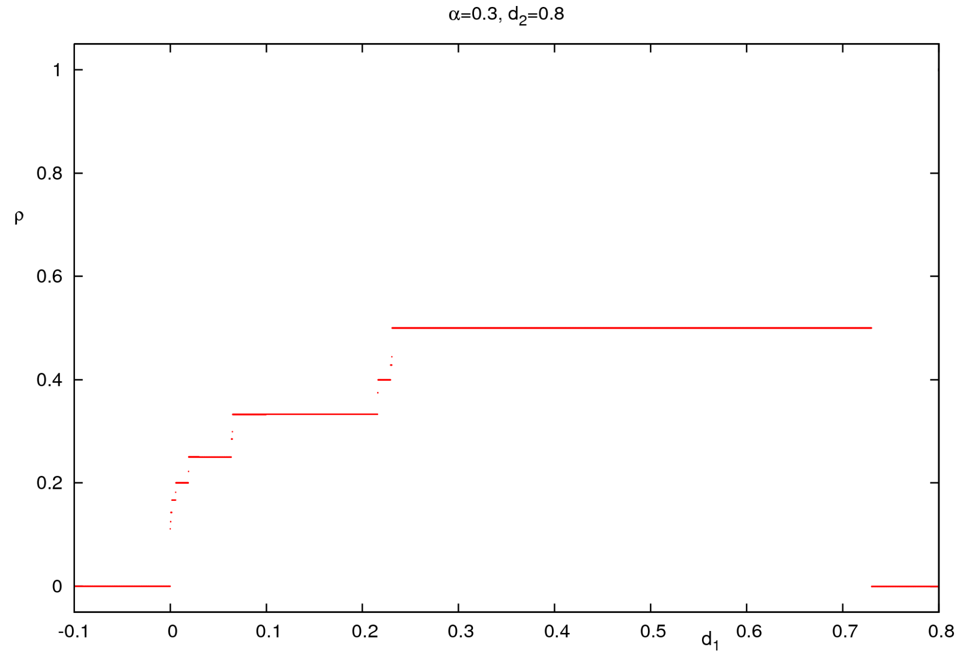




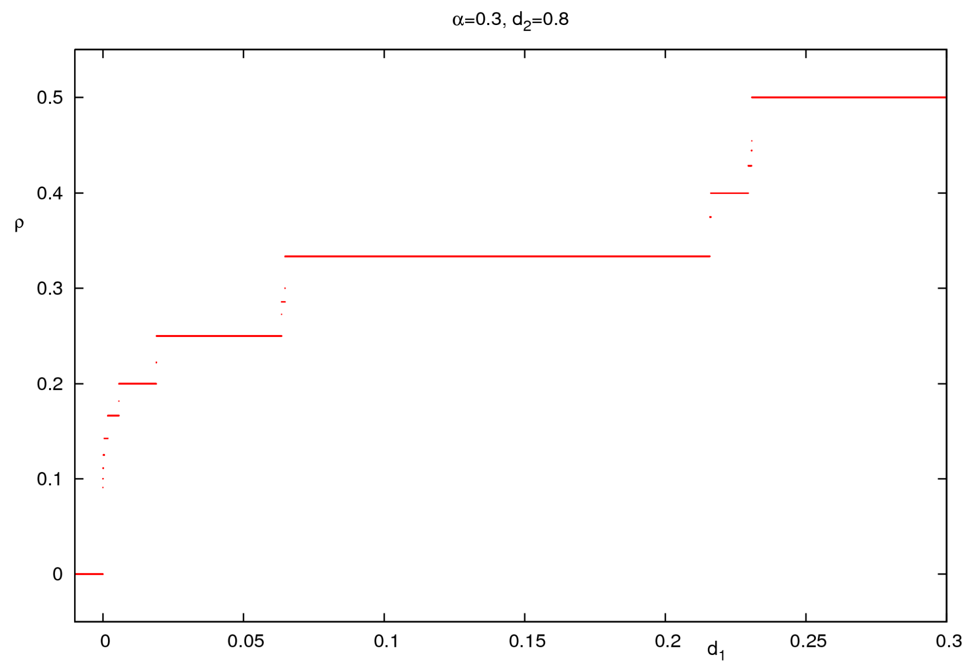
This figure illustrates how the periodicity regions change with α for Cases C. ($d_2 = 0.8$).



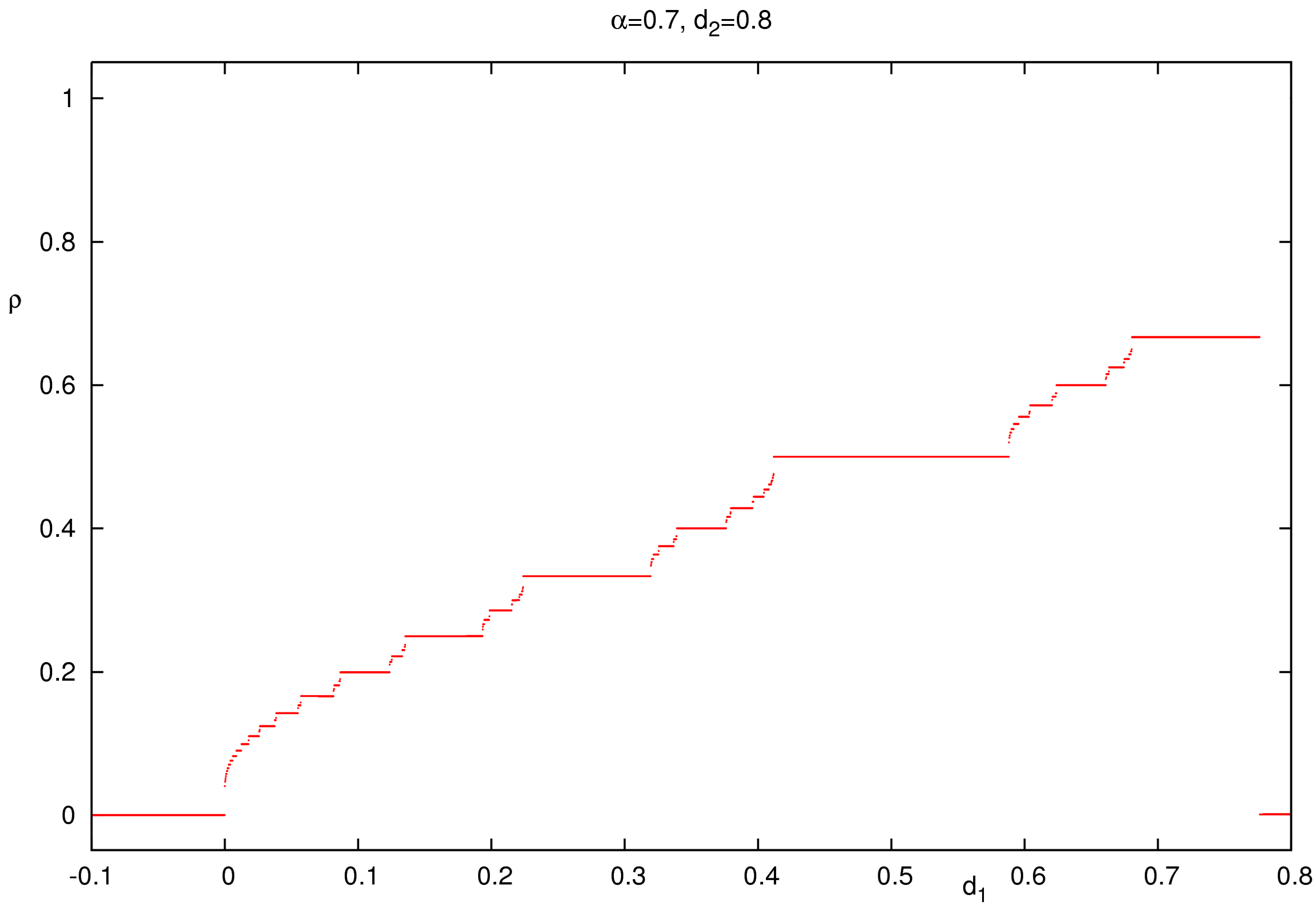
Here's the rotation number for Case C ($\alpha = 0.3;\ d_2 = 0.8$).



Here's its blow-up.



Here's the rotation number for Case C ($\alpha = 0.7;\ d_2 = 0.8$).



5. Some Concluding Remarks

- A regime-switching model of credit frictions, by Matsuyama (2007), can display a wide array of dynamical behavior.
- This paper showed that a complete characterization of the dynamic behavior on the parameter space is feasible for a PWL case. Among others, it showed:
  - How stable cycles of any integer period can emerge.
  - Along each stable cycle, how the economy alternates between the expansionary and contractionary phases.
  - How asymmetry of cycles (the fraction of time the economy is in the expansionary phase) varies with the credit frictions parameters.
  - How the economy may fluctuate for a long time at a lower level before successfully escaping from the poverty, etc.
- The analysis was done for a restrictive set of assumptions (2 projects with 2 switching points), because it creates a rich array of dynamics with a relatively few parameters. With more projects, more switching points, generating even richer behaviors.
- The discontinuity and piecewise linearity simplify the analysis. Similar results can be numerically obtained when the discontinuous, piecewise map is approximated by a continuous map with very steep slopes.
- More generally, the analytical tool used in this paper should be useful for many other dynamic economic models.